CHAPTER 1

Logic

Logic is correct thinking. In a wide variety of circumstances, without any formal training, we are able to do a good job of thinking over a range of complex problems.

So why study logic formally? First, there are systematic deficiencies in our untrained cognitive behavior that can to some extent be remedied by becoming aware of the deficiencies and aware of how we should in fact be reasoning. We will discuss many examples of this. Second, logic is part of the foundation of more advanced formal reasoning like mathematics. As we will see, logic provides us with a clear notion of proof. Having a rigorous concept of proof is critical to the advancement of mathematics. See Proofs and Refutations by Imre Lakatos, for a good historical example of how mathematicians struggled to prove Euler’s conjecture because they didn’t have a satisfactory notion of proof.

1. Arguments

An argument is attempt to convince someone of something. Arguments are attempts to give the listener or reader of the argument reasons for believing that some particular claim is true.

(1) There have to be reasons.
(2) There has to be some sort of presumption that the listener or reader does not already believe the particular claim.
(3) The reasons have to be intended for the listener or reader to come to believe the particular claim. They don’t have to explain why the claim is true; they just have to give some reason why one should believe it.

How can you recognize an argument? One clue is that they have expressions like ‘because’, ‘since’, ‘for’, ‘suppose’, ‘assume’, ‘therefore’, ‘hence’, ‘thus’, ‘it follows that’, ‘we can conclude that’, ‘for the following reasons’. But the clues are only a very rough guide and are often misleading. The definitive way to tell is to take all the clues available from the source and the context and to answer the question, “Does it give me reasons for adopting belief in some claim?”

1.1. The Difference between Arguments and Explanations. Often it is hard to tell the difference between an argument and an explanation because both use many of the key words like ‘because’, ‘since’, and ‘so’.

Whether a written or spoken passage is an argument is logically independent of whether it is an explanation. This means: An argument might be an explanation and it might not.

An explanation might be an argument and it might not.

An explanation tells you why something is true.
An argument tells you why you should believe it.
Here is an explanation that is not an argument: The sky is blue because the atmosphere is mainly composed of nitrogen, and nitrogen scatters blue light over a much wider range of angles than other colors.

Here is an argument that is not an explanation: The sky is really red tonight; I’ve just been watching it.

Here is an argument that is an explanation: The sky is usually clear where I live because I live on the east edge of the Rocky Mountains.

Here is a passage that is neither an argument nor an explanation: The sky is cloudy. It gets cloudy here every now and then.

(1) Fats impair immune function. In both in-vitro tests and experiments using intravenous soybean oil infusions in volunteers, fats reduce the vigilance of white blood cells. Researchers in New York put healthy volunteers on diets that reduced fat content to 20 percent of calories. Three months later, blood samples showed that natural killer cell activity was greatly improved. Not surprisingly, vegetarians have been shown to have more than double the natural kill cell ability, compared to non-vegetarians.

(2) Among the many ways the pastoralists interact with and represent their aural environment, one stands out for its sheer ingenuity: a remarkable singing technique in which a single vocalist produces two distinct tones simultaneously. One tone is a low, sustained fundamental pitch, similar to the drone of a bagpipe. The second is a series of flutelike harmonics, which resonate high above the drone and may be musically stylized to represent such sounds as the whistle of a bird, the syncopated rhythms of a mountain stream or the lilt of a cantering horse. (Scientific American, September 1999)

(3) Under certain conditions, cow’s milk proteins pass through the gut into the bloodstream, eliciting the production of antibodies. These antibodies end up attacking not only the milk proteins but also pancreatic beta-cell proteins that happen to be structurally similar to those in cow’s milk. Viral infections cause these beta-cell proteins to be exposed to the antibodies. During viral infections over the next several years, intermittent antibody attacks gradually destroy the beta cells. In late childhood or early adulthood, insulin levels are so low that diabetes becomes manifest. (Karjalainen J, Martin JM, Knip M, et al. A bovine albumin peptide as a possible trigger of insulin-dependent diabetes mellitus. N Engl J Med 1992;327:302-7.)

(4) The idea that there is an epidemic of human cancer caused by synthetic industrial chemicals is false. Linear extrapolation from the near-toxic doses in rodents to low-level exposure in humans has led to grossly exaggerated mortality forecasts. Such extrapolations cannot be justified by epidemiological evidence. Furthermore, relying on such extrapolations for synthetic chemicals while ignoring the enormous natural background leads to an unbalanced perception of hazard and allocation of resources. Far from being the source of unhealthiness, the progress of scientific research and technology by industry will continue to lengthen human life expectancy; indeed, there has been a steady rise in life expectancy in the developed countries.
(5) It is one of the clearest symptoms of the decadence besetting the academy that the ideals that once informed the humanities have been corrupted, willfully misunderstood, or simply ignored by the new sophistries that have triumphed on our campuses. We know something is gravely amiss when teachers of the humanities confess—or, as is more often the case, when they boast—that they are no longer able to distinguish between truth and falsity. We know when something is wrong when scholars assure us—and their pupils—that there is no essential difference between the disinterested pursuit of knowledge and partisan proselytizing, or when academic literary critics abandon the effort to identify works of lasting achievement as a reactionary enterprise unworthy of their calling. And indeed, the most troubling development of all is that such contentions are no longer the exceptional pronouncements of a radical elite, but have increasingly become the conventional wisdom in humanities departments of our major colleges and universities.

(6) After spending more than $40 million on the investigation, the FBI and the National Transportation Safety Board (NTSB) have not found a definitive answer for why the center fuel tank exploded. Yet they have ruled out a missile as the cause. The NTSB believes an undetermined system flaw produced an electrical spark that ignited jet fuel vapors in the tank. Prior to the official embrace of this mechanical explanation, the missile expert was among several scientists invited by FBI agents to explore the missile theory. He was made privy to evidence suggesting that TWA 800 could have been shot down, consisting of eyewitness accounts of a “flare-like object” shooting skyward moments before the plane exploded. Later he examined the debris in the Calverton hangar. The missile expert has also been in contact with military labs where, he says, the chemists have been unable to make jet fuel vapor explode as the NTSB says it did in TWA 800’s center fuel tank. “The labs told the NTSB there’s a big problem—it can’t happen.” The NTSB wouldn’t listen. He says, “They were adamant that [the labs] had to find something.” Davey, R.; Village Voice; July 14-20, 1999.

(7) Today’s public relations (PR) industry is related to democracy in the same way that prostitution is related to sex. When practiced voluntarily for love, both can exemplify human communications at its best. When they are bought and sold, however, they are transformed into something hidden and sordid. There is nothing wrong with many of the techniques used by the PR industry—lobbying, grassroots organizing, using the news media to put ideas before the public. As individuals, we not only have the right to engage in these activities, we have a responsibility to participate in the decisions that shape our society and our lives. Ordinary citizens have the right to organize for social change—better working conditions, health care, fair prices for family farmers, safe food, freedom from toxins, social justice, a humane foreign policy. But ordinary citizens cannot afford the multi-million dollar campaigns that PR firms undertake on the behalf of their special interest clients, usually large corporations, business associations and governments. Raw money enables the PR industry to
mobilize private detectives, attorneys, broadcast faxes, satellite feeds, sophisticated information systems and other expensive, high-tech resources to outmaneuver, overpower and outlast citizen reformers.

(8) In rather the same way as new movies are now ‘reviewed’ in terms of their first weekend gross, new candidates have become subject to evaluation by the dimensions of their ‘war chest.’ This silly, archaic expression defines other equally vapid terms like ‘credibility’ and ‘electability’ and ‘name recognition,’ which become subliminally attached to it. In many cases the crude cash-flow measure is as useful in deciding on a politician as it is in making a choice at the multiplex; you might as well see the worthless movie that everyone else has seen, or express an interest in the unbearably light ‘front runner,’ so as not to be left out of the national ‘conversation.’ (Hitchens C, The Nation, August 23/30 1999)

(9) Secondly, we must proceed to a definition of the term ‘art’. This comes second, and not first, because no one can even try to define a term until he has settled in his own mind a definite usage of it: no one can define a term in common use until he has satisfied himself that his personal usage of it harmonizes with the common usage. Definition necessarily means defining one thing in terms of something else; therefore, in order to define any given thing, one must have in one’s head not only a clear idea of the thing to be defined, but an equally clear idea of all the other things by reference to which one defines it.

(10) One of the most cherished freedoms in a democracy is the right to freely participate in the “marketplace of ideas.” We value this freedom because without it, all our other freedoms are impossible to defend. In a democracy every idea, no matter how absurd or offensive, is allowed to compete freely for our attention and acceptance. Turn on the TV, and you’ll find plenty of absurd and offensive examples of this principle in action. On the Sunday public affairs shows you’ll find Republicans, Democrats, Republicans who love too much, and Democrats who love Republicans. On “A Current Affair” or “Oprah Winfrey,” you’ll find self-proclaimed werewolves, worshippers of Madonna, and doomsday prophets from the lunatic fringes of American society.

(11) The curse of having too many recreational options, of course, is further encouraged by the Dominican Republic’s substantial development of large modern, luxurious all-inclusive resorts, scattered up and down both coastlines. Most offer nearly all watersports (from boardsailers and sailboats to wave runners and scuba diving), meals, alcoholic and non-alcoholic beverages, entertainment and children’s daycare for incredibly low package rates. Couple this with the unusually low cost and affordable rates most airlines provide to the island and the Dominican Republic is perhaps one of the Caribbean’s best kept-secrets.

(12) According to David Hume, causes are sufficient conditions for their effects: “We may define a cause to be an object, followed by another, and where all the objects similar to the first, are followed by objects similar to the second.” (1748, section VII.) Later writers refined Hume’s theory, but still characterized the causal relation in terms of necessary and sufficient conditions. One of the best known approaches is Mackie’s theory of inus
conditions. An inus condition for some effect is an insufficient but non-redundant part of an unnecessary but sufficient condition. Suppose, for example, that a lit match causes a forest fire. The lighting of the match, by itself, is not sufficient; many matches are lit without ensuing forest fires. The lit match is, however, a part of some constellation of conditions that are jointly sufficient for the fire. Moreover, given that this set of conditions occurred, rather than some other set sufficient for fire, the lighting of the match was necessary: without it, the fire would not have occurred.

(13) Beliefs produced by visual experience are in large part selfascriptive: the subject believes not only that the world is a certain way but also that he himself is situated in the world in a certain way. To believe that the scene before my eyes is stormy is the same as to believe that I am facing a stormy part of the world. Elsewhere I have argued that the objects of such beliefs should be taken, and that the objects of all beliefs may be taken, as properties which the subject self-ascribes. Hence the content of visual experience likewise consists of properties—properties which the subject will self-ascribe if the visual experience produces its characteristic sort of belief.

(14) Video poker began as an afterthought, but as profits surged through the 1980’s, operators longed for legitimacy. South Carolina law forbade games of chance, so at first the operators relied on linguistic chicanery: Poker, they said, equals pinball. Play pinball well enough and you win a free game. Same thing in video poker. Suppose it costs twenty-five cents to play a game of pinball or poker. Well, then each skillfully won free game must be worth twenty-five cents. And if each game is worth twenty-five cents, surely you ought to be able to collect a quarter for it. And if you can collect a quarter for one free game, surely you should be able to collect 4,000 quarters for drawing a royal flush. (David Plotz, Harpers, August 1999, p.66)

(15) The arrow of time is one of the deepest paradoxes of conventional physics today. According to all the laws of physics there should be no distinction between past and future, no direction to time. Since the second law of thermodynamics says that entropy necessarily increases with time, and thus the past and future differ, the second law, too, is contradicted.

(16) As new research is pushing forward the day the planet became habitable, other discoveries are pushing back the first signs of life. Microfossils found in ancient rocks from Australia and South Africa demonstrate that terrestrial life was certainly flourishing by 3.5 billion years ago. Even older rocks from Greenland, 3.9 billion years old, contain isotopic fingerprints of carbon that could have belonged only to a living organism. In other words, only 100 million years or so after the earliest possible point when Earth could have safely supported life, organisms were already well enough established that evidence of them remains today. This narrowing window of time for life to have emerged implies that the process might have required help from space molecules. (Scientific American, July 1999)

(17) On September 21, 1993, President Clinton signed into law the National and Community Service Trust Act, which appropriated $1.5 billion for an embryonic national service program. The Clinton national service plan
will create 20,000 federally funded, full-time youth jobs this fall. That number will increase to 33,000 in 1995 and 47,000 in 1996, for a total of 100,000 participants. The lure for enrollment in the program consists of a $4,725 college scholarship for each year of service. The CNCS did not spring full-blown from the brow of Bill Clinton. The corporation is an enhancement of the 1990 National and Community Services Act, through which the states were encouraged to develop federally approved guidelines for “volunteer” community service through public schools and other agencies. As he signed the 1990 legislation, then-President George Bush declared, “From now on, any definition of a successful life must include serving others.” The 1990 act earmarked $25 million for the creation of “peer-to-peer pressure groups” intended to goad reluctant citizens to “volunteer” for service . . . . However, Americans continue to be the world’s most devoted altruists, donating considerable volunteer service without any prompting from government; this has been acknowledged by Eli Segal. In a speech made last November 8th, Segal pointed out that “94 million American citizens right now are doing volunteer work. Sixty percent of young people between 12 and 17 are now volunteering over three hours a week.” The spontaneous private response to recent disasters, such as the Midwest flood and the California earthquake, illustrate that Americans are capable of addressing community needs without government intrusion. (Grigg, W.N.; The New American; Vol. 10, No. 06, March 21, 1994)

2. Sentences and Statements

There are many kinds of sentences:

(1) You’re in danger.
(2) Who’s there?
(3) Stop!
(4) I’d like a snack.
(5) Please.
(6) Silence is golden.
(7) Free your slaves, or else!
(8) I don’t know.
(9) Vermont used to be a relatively fun place to go skiing, even if it isn’t the cheapest site around.
(10) Huh?
(11) If I don’t steal his watch, I doubt someone else will.

Any sentence of which it makes sense to say that it is true or it is false is called a statement. Statements are declarative sentences; they make some claim.
The statements are 1, 4, 6, 8, 9, 11. The others are not.

3. Answers to the Identification of Arguments

‘A’ means it is an argument, ‘N’ means it is not.
4. Propositions

Werner said, “Ich hoffe du bist nicht krank.” I report that Werner stated that he hopes you aren’t sick. The content of Werner’s claim is that he hopes you aren’t sick. The two statements, “Ich hoffe du bist nicht krank,” and “I hope you aren’t sick,” are two different sentences and two different statements that express the same content.

A proposition is the content of a statement. For example, Werner was expressing the proposition that he hopes you aren’t sick. A proposition is not a language dependent entity, although it might be that some statements in some languages pick out propositions that cannot be (simply) expressed in other languages.

5. Truth

In classical logic, every statement has exactly one of the following two truth values: true or false.

There are no cases where it is both true and false. There are no cases where it fails to have a truth value. There are no in-betweens. There are no degrees of truth where for example we say something is mostly true or 98% true.

We will work with these assumptions throughout our discussion. In more advanced logic, we can relax some of these requirements and explore the philosophical consequences, but for now, we’ll leave these issues to folks with a lot of time on their hands.

6. Vagueness

How do we evaluate whether a statement is true or false if it isn’t clear cut? We assume that there are some standards for judging that are as precise as one needs, and we evaluate the statement relative to one of these standards.

Example 1. “Madonna is old.” “White is a color.”

For the purposes of evaluating arguments, we almost never need to say that one standard is wrong and another one is right. We will almost always be able to say, you can go either way: it’s either true or false depending on how you look at it.

7. Possibility

When a statement is possibly true, we say the statement is called consistent. When a statement is not possibly true, we say the statement is inconsistent, or we say it’s a contradiction.

Example 2. Consistent statements

“There are only 3 planets in the Solar system.”
“Copper is metallic.”

We often say things like “I can’t speak Portuguese,” or “Birds can’t fly to the moon.” Yet statements that seem literally true in one sense, can be false in another sense.

In logic, we focus on the broadest sense of possibility. We take nothing for granted except being logically consistent. If we can coherently imagine it, it is possible.
I can coherently imagine that birds can fly to the moon, so it’s logically possible for birds to fly to the moon.

**Example 3. Inconsistent statements**

- There are square circles.
- Colorless green bricks exist.
- Some mothers have had no children.
- Joe is from Manhattan, and he isn’t from Manhattan.
- Fred Flintstone is just a fictional character, yet I know that he is a real live person.

Rules for judging possibility: If you can imagine it, it’s possible, but you have to be imagining correctly (i.e. coherently).

In judging possibility, we adopt a convention where you have to be consistent in your standards for resolving vagueness. If the sentence has the word ‘blue’ at the beginning and ‘blue’ at the end, you have to have the same standards for determining what counts as blue.

You have to stick to the meanings that are given in natural language (like English). You can’t say that it’s possible for there to be round squares because of the fact that we could make up a language where the word ‘round’ means pepperoni and the word ‘squares’ means pizzas. Stick to what the English words mean, and then look for flexibility that allows the sentence to be true.

You cannot coherently imagine a mathematical or logical contradiction. You may think that you can imagine the following equation is true, but you cannot coherently imagine it:

\[(129442093) + (390 \times 1023) - 389920 = 399 \times (890013) - 219\]

8. Subtleties with Contradictions

There are a number of sentences that look like contradictions, but are not really contradictions in the strictest sense. Here are several that look contradictory but are not, or are at least not clearly contradictory.

1. This statement is not true.
2. This sentence is not written in English.
3. This flower is a peony, but I don’t know whether it’s a peony.
4. The word ‘horses’ does not refer to horses.
5. Nothing exists.
6. Every sentence is true.
7. If the coin is made of gold, then it isn’t.
8. Everyone is mortal, and everyone is immortal.
9. There exists a blue object that has no physical extension.

The important lesson at this point is just that there are some tricky cases that we will need to address later on. You do not need to be able to explain why every one of the above statements is not a contradiction.

A final bit of advice for evaluating whether a statement is a contradiction: Do not worry about any seeming conflicts that arise in virtue of the *production* of the statement, i.e. how (or the mere fact that) the statement is spoken or printed or thought. You will not be tested on any self-referential conflicts because those are too difficult to address using the resources of introductory logic.
9. Necessity

A statement that cannot be false is called necessarily true or just necessary, and we also call it a tautology.

A statement that can be true and can be false is called contingent.

Exercise 1. Label each of the following as being necessary, contingent, or inconsistent

1. Suddenly, all the stars in the nighttime sky disappeared, leaving only the moon visible to the naked eye or the astronomer’s telescope.
2. The princess kissed the frog, and it turned into a handsome goat.
3. No human beings exist.
4. There is a human being taller than every human being.
5. The triangle had four sides, two of which were parallel.
6. Napoleon proved that the Earth is flat.
7. The astronaut lost his helmet during a space walk, but he survived just fine without any air to breathe.
8. The daughter of my brother is my sister’s nephew.
9. Each point on a circle is equidistant from the center.
10. Ashley was unemotional yet at the same time enraged.
11. Godzilla stomped all over Tokyo, and then he turned himself into a delicious slice of cheesecake.
12. I drink, but I do not exist.
13. The archbishop was assassinated, but he hasn’t died yet.
14. There is a pain in my leg that I can’t feel.
15. She saw the invisible wizard.
16. Kendrick saw the tiger escape the cage, but it didn’t really escape.

For all of the above statements, you ought to be able to identify whether it is necessary, contingent, or inconsistent.

Summary of definitions:

- A necessary statement, or tautology, is a statement that cannot possibly be false.
- A contradiction is a statement that cannot possibly be true.
- A contingent statement is one that is possibly true and possibly false.
- A set of statements is consistent if they can all be true together, i.e. if there is at least one possibility where all the statements are true.
- A set of statements is inconsistent if they cannot all be true together.

10. Formal Arguments

Formally, arguments are a finite number of statements, which we call premises, plus a single statement, which we call the conclusion.

We write arguments in the following way:

It never matters what order the premises are in.

From here on, anytime we have a collection of statements that is organized in a vertical list with a single statement under the horizontal bar, it is by stipulation an argument regardless of whether it seems like the premises are a good reason to believe the conclusion. This allows us to ignore context and the blurry distinction between arguments and non-arguments.
11. Counterexample

A **counterexample** to an argument is a possible situation where all the premises are true and the conclusion is false.

\[
\text{Premise 1} \\
\text{Premise 2} \\
\text{Premise 3} \\
\ldots \\
\text{Premise 62} \\
\text{Conclusion}
\]

**All reptiles are green.**

**None of the animals in the zoo are reptiles.**

**There are no green animals in the zoo.**

The counterexample must be a fully fleshed out situation where it is clear that all reptiles are green and that none of the animals in the zoo are reptiles while at the same time there are some green animals (or at least one) in the zoo.

**Exercise 2.** Describe counterexamples to the following arguments

Lenny has been coughing a lot lately.

Lenny is sick.

Charlotte has 3 children.

Charlotte has 3 daughters.

Frogs are aquatic.

Anything that lives in a pond or river is aquatic.

Frogs live in ponds or rivers.

12. Validity

A **valid argument** is an argument that has no counterexamples.

An **invalid argument** is an argument that’s not valid.

**VALID** means **COUNTEREXAMPLE-FREE**

Note that it does not make sense to say a statement is valid. Validity is an attribute of arguments only. Individual statements can be true or false, but they cannot be valid or invalid.

13. Validity Practice Problems

**Exercise 3.** Identify which of the following arguments are valid.

All philosophers are nerds.

Oscar is a philosopher.

Oscar is a nerd.

No creatures live on the moon.

The moon is habitable.

Some cats are white.

Fluffy is a cat.

Fluffy is white.

Bob has a red shirt.

Bob has a shirt that is colored.
Seth is unemployed.
Seth does not have a regular paying job.
All drummers own some kind of drum kit.
Jim owns his own drum kit.
Jim is a drummer.
Radishes are partially red.
Radishes are partially green.
Radishes are partially colored.
Oscar owns 3 Tool shirts.
3 Tool shirts belong to Oscar.
Only people do social work.
All social work is done by people.
No one who studied philosophy would have done poorly on the LSAT.
John is a person.
John did poorly on the LSAT.
John did not study philosophy.
Anyone who smokes will get lung cancer.
Trevor smokes.
Trevor will get cancer.
Not everyone who studies will get an A.
Lisa did not study.
Lisa will not get an A.
Amy had $300 in her piggy bank yesterday.
No one has come near her piggy bank within the last few days.
Amy still has $300 in her piggy bank.
Rhonda was born exactly 13 years ago.
Rhonda is alive today.
Rhonda has been alive for 13 years.
President Sadat was murdered.
President Sadat died.
New Delhi is a part of India.
India is a part of Asia.
New Delhi is a part of Asia.
Seth’s brother is Steve.
Seth’s son is John.
John is Steve’s nephew.
$200 is too much to pay for an old laptop.
Jerry paid $160 for his old laptop.
Jerry didn’t pay too much for his laptop.
All fruit trees produce flowers sometime during the year.
If I own a fruit tree, it will produce flowers sometime.
Paul either got a speeding ticket or he spent time in jail for reckless driving.
It turns out that Paul didn’t spend any time in jail.
Thus, Paul got a speeding ticket.
Yesterday, Gus loaded up his truck with a load of manure at the feed lot, drove it home, dumped the whole load in his garden, and then started the hard work of tilling it into the soil.

Thus, Gus drove his truck yesterday.

One of the cattle got out of the pen yesterday afternoon.

No one at the ranch went out to find the cow and put her back in the pen.

Thus, there is one less cow in the pen this morning than there was yesterday morning.

All children like to eat cookies.

Amber Evans likes to eat cookies.

Thus, Amber Evans is a child.

The Colorado River runs through the Grand Canyon.

Kelli Hilgenfeld is rafting in the Colorado River.

Thus, Kelli Hilgenfeld is in the Grand Canyon.

All clowns are easily distracted by booze.

Thus, circus elephants are the second-most entertaining animals, after the “dancing” bears.

All brown things come from the soil, but no lentils come from the soil.

Thus, lentils are never brown.

Horses can be ridden with a western saddle.

Thus, horses can be ridden.

It’s not true that every flower is blue.

Thus, there is at least one blue flower.

Jane has not stopped robbing banks.

Jane is a bank robber.

Tulips grow in the Netherlands.

The Netherlands is in Europe.

Thus, tulips grow in Europe.

All things that have motors require oil.

All cars require oil.

Thus, all cars have motors.

All restaurants serve apple pie.

No restaurants serve ice cream.

Matilda’s is a restaurant.

Matilda’s serves at least one of the following: apple pie or ice cream.

Roses are never red.

Roses are never white.

Thus, it is untrue that roses are neither red nor white.

The only way to pass the exam is to study at least 10 hours.

George studied 20 hours.

George will pass the exam.

The sun rose today.

The sun rose yesterday.

In fact, there has not been a day yet, in which the sun has not risen.

Therefore, the sun will rise tomorrow.
Aaron Burr shot Alexander Hamilton in a duel.
Alexander Hamilton was a US statesman.
Therefore, at least one US statesman has been shot in a duel.
All nice people are saints.
Some do-gooders are not nice people.
Thus, some do-gooders are not saints.
All pleasant people have comfortable chairs.
Some comfortable chairs are expensive.
Some pleasant people have expensive chairs.
All citizens who are not traitors are present.
All officials are citizens.
Some officials are not present.
Thus, there are traitors present.
Only pacifists are Quakers.
There are religious Quakers.
Thus, some pacifists are religious.
No violinists are not wealthy.
There are no wealthy cellists.
Thus, no violinists are cellists.
No candidate who is endorsed by labor can carry the farm vote.
No one can be elected unless he carries the farm vote.
No candidate endorsed by labor can be elected.
Oscar likes Tool.
Tool plays alt-metal music.
Oscar likes alt-metal music.
Alvin is taller than Bianca.
Bianca is taller than Chandler.
Diana is shorter than Chandler.
Therefore, Diana is shorter than Alvin.
Nigel borrowed Natasha’s vacuum cleaner.
Nigel never returns what he borrows.
Therefore, Nigel still has Natasha’s vacuum cleaner.

Here are some trickier arguments that employ some subtleties....

Dancing is popular in Turkmenistan.
Turkmenistan is in Asia.
Dancing is popular in Asia.
All actors are self-centered.
Groucho Marx was an actor.
Groucho Marx was self-centered.
The deadliest fish in the ocean is Oscar, the buck-toothed gar.
Sharks are not the deadliest fish in the ocean.
Doctors and lawyers are professionals.
Professionals and executives are respected.
Doctors are respected.

The answers to these practice problems are located in §17.
14. Summary of Definitions

- A **statement** is the proposition (or content) of a sentence that makes a claim.
- A **proposition** is the content of a statement, what the statement claims.
- A **necessary** proposition, or **tautology**, is a proposition that cannot possibly be false.
- A **contradiction** is a proposition that cannot possibly be true.
- A **contingent** proposition is one that is possibly true and possibly false.
- A set of propositions is **consistent** if they can all be true together, i.e. if there is at least one possibility where all the propositions are true.
- A set of propositions is **inconsistent** if they cannot all be true together.
- An argument is a set of propositions (called **premises**) and a proposition (called a **conclusion**).
- A **counterexample** to an argument is a possible situation where the premises are all true and the conclusion is false.
- An argument is **valid** if it has no counterexamples.
- An argument is **invalid** if it has at least one counterexample.

15. Why Logic is Tough

Your innate sense of which arguments are good and which ones are bad do not correspond with what logic says is valid and invalid. For instance, your brain is set up to think using important background information and to ignore the kind of outlandish possibilities that you need to consider when checking validity. The good news is that your informal reasoning ability is very good at most tasks, and with many tasks your innate ability surpasses the ability of any computer using the formal methods we will learn. The bad news is that many times our intuitive reasoning ability misleads us, including situations that are very important to us, like whether to have a risky surgery, how to design a multi-million dollar experiment.

The point to keep in mind is that there are tools of logic that can overcome the failings of our intuitive logic. The most difficult thing for you as a student of logic is to remember to use the rules. It is very tempting to take shortcuts, but dangerous as well.

15.1. Why Should We Care about Validity? In diagnosing a real argument, it is very useful to distinguish the strength of an argument’s reasoning from the strength of its premises. An argument might start off making assumptions that everyone agrees are reasonable and well founded but then draw conclusions through fallacious reasoning. An argument might also use impeccable reasoning, but nevertheless be faulty because it starts off with outlandish assumptions.

Validity is the ultimate in **reasoning strength**. It gives us a guarantee that the conclusion is true if the premises are. The ultimate in **premise strength** is truth, although it often happens that we do not have reliable access to the truth. Thus, we often measure the strength of the premises in terms of their plausibility, which quantifies how likely a statement is to be true given all we know. Together, truth of the premises, and validity of the argument, constitute the ultimate in argument strength because they guarantee that the conclusion is true.

A valid argument is good because it gives us a 100% foolproof guarantee that if the premises are true, then the conclusion is also true.
There is one and only one burglar.
Only the burglar could have left the glass-cutter in the vault.
Gonzo the night watchman is the only person who could have left a glass-cutter in the vault.
Thus, Gonzo is the burglar.

Because this argument is valid, one way to prove that Gonzo burgled is to prove that each of the three premises is true. Once each of the premises has been proven true, logic dictates that Gonzo is the guilty party.

Likewise, if we can identify an argument as having poor reasoning, we know that it will be a waste of time to investigate whether the premises are true.

Our experiment showed that the people we tested who had heart attacks had higher IQ’s than people we tested who didn’t.
If IQ’s don’t have anything to do with heart health, then there is only a 0.4% chance that we would have gotten the results we got.
Thus, there is a 99.6% chance that people who have had heart attacks have higher IQ’s than people who didn’t have heart attacks.

Because this argument is invalid, we should not (on the basis of this argument) waste our time and money trying to repeat the experiment to verify the results.

15.2. VALID ARGUMENT ≠ GOOD ARGUMENT. An argument that is valid has superb reasoning strength, but that doesn’t mean the argument is good. The overall argument strength also depends on whether the premises are true and several other factors we have not discussed.

Here is a valid argument that is not a good argument for two reasons:

Trees never need water to grow.
Thus, trees never need water to grow.

15.3. INVALID ARGUMENT ≠ BAD ARGUMENT. An invalid argument fails to give us a guarantee that the conclusion will be true when the premises are true, but sometimes we don’t need an absolute guarantee. Almost all real arguments are invalid arguments.

Here is an invalid argument that is a pretty good argument:

Every emerald that has been found so far has been green.
Thus, the next emerald someone finds will be green.

15.4. Conversational Implicature vs. Logical Implication. Suppose someone says, “My preacher hasn’t been drunk at all for this entire week!” What is the reasonable thing to infer?

One reasonable thing to infer is something like, “My preacher is often drunk.”

This is not a logical inference because

My preacher hasn’t been drunk at all for this entire week.
My preacher is often drunk.

is invalid. We call this kind of inference ‘conversational’ because the reason we find it plausible has to do with accepted rules for having conversations.

If my preacher seldom or never drinks, I wouldn’t have any reason to mention that he hasn’t been drunk for the past week. Therefore, when I explicitly mention the past week, you are led to think that there must be a good reason for my specifying his being sober specifically during the past week. The only good reason
that comes to mind off-hand for my mentioning it seems to be that it’s unusual, that he is normally drunk.

The way to tell a logical inference from a conversational inference is to see whether you can deny the inference (cancel it) without retracting or contradicting what you originally said.

I can say, “My preacher hasn’t been drunk this entire week. In fact, he never drinks.” without having to retract anything.

I can’t say, “My notebook is red, but it doesn’t have any color.” If I say it doesn’t have any color, I am effectively retracting my comment that it is red.

16. EX FALSO QUODLIBET

This latin phrase means, “From an absurdity, anything follows.” It is a consequence of the definition of validity is that if the premises are contradictory, the argument is valid, no matter what the conclusion is. Here’s why:

(1) Assume we have some inconsistent premises.
(2) What it means for the premises to be inconsistent is that it is impossible for the premises to be all true.
(3) If it is impossible for the premises to be all true, then it is impossible for the premises to be all true while the conclusion is false.
(4) That implies there are no counterexamples.
(5) That implies the argument is valid.

Godzilla does not really exist.
Geoff is not tall.
Geoff is tall.

Thus, Godzilla really exists.

Everest is the tallest mountain on Earth.
K2, a mountain on Earth, is taller than Everest.

Thus, eagles like to swim underwater with loons and ducks.

1 + 1 = 3
Thus, the moon is made of cheese.

Jennifer both is and is not the Queen of the Pecan Festival.

Thus, 23 - 7 = 11

It is another consequence of the definition of validity that if the conclusion is tautologous, the argument is valid. Here’s why:

(1) Assume we have some tautologous conclusion.
(2) What it means for the conclusion to be tautologous is that it is impossible for the conclusion to be false.
(3) If it is impossible for the conclusion to be false, then it is impossible for the conclusion to be false while the premises are all true.
(4) That implies there are no counterexamples.
(5) That implies the argument is valid.

Trampolines are fun.

Thus, red is not the same color as blue.

The governor of Alabama got caught engaging in bizarre sex acts.

Thus, the area of a circle is pi times the radius squared.
17. Answers to Informal Validity Practice Problems

(1) All philosophers are nerds... V
(2) No creatures live on the moon... I
(3) Some cats are white... I
(4) Bob has a red shirt... V
(5) Seth is unemployed... V
(6) All drummers own some kind of drum kit... I
(7) Radishes are partially red... V
(8) Oscar owns 3 Tool shirts... V
(9) Only people do social work... V
(10) No one who studied philosophy... V
(11) Anyone who smokes will get lung cancer... V
(12) Not everyone who studies will get an A... I
(13) Amy had $300 in her piggy bank yesterday... I
(14) Rhonda was born exactly 13 years ago... I
(15) President Sadat was murdered... V
(16) New Delhi is a part of India... V
(17) Seth’s brother is Steve... V
(18) $200 is too much to pay for an old laptop... I
(19) All fruit trees produce flowers sometime during the year... V
(20) Paul either got a speeding ticket... V
(21) Yesterday, Gus loaded up his truck... V
(22) One of the cattle got out of the pen... I
(23) All children like to eat cookies... I
(24) The Colorado River runs through the Grand Canyon... I
(25) All clowns are easily distracted by booze... I
(26) All brown things come from the soil... V
(27) Horses can be ridden with a western saddle... V
(28) It’s not true that every flower is blue... I
(29) Jane has not stopped robbing banks... I
(30) Tulips grow in the Netherlands... V
(31) All things that have motors require oil... I
(32) All restaurants serve apple pie... V
(33) Roses are never red... I
(34) The only way to pass the exam... I
(35) The sun rose today... I
(36) Aaron Burr shot Alexander Hamilton... V
(37) All nice people are saints... I
(38) All pleasant people have comfortable chairs... I
(39) All citizens who are not traitors are present... I
(40) Only pacifists are Quakers... V
(41) No violinists are not wealthy... V
(42) No candidate who is endorsed by labor... V
(43) Oscar likes Tool... I
(44) Alvin is taller than Bianca... V
(45) Nigel borrowed Natasha’s vacuum cleaner... I
(46) Dancing is popular in Turkmenistan... I (The natural interpretation of ‘X is popular in Y’ is not ‘there is some region within Y where X is popular.’ Instead, it means ‘throughout Y as a whole, X is popular.’)

(47) All actors are self-centered... V (If we interpret ‘All actors are self-centered in a tenseless way, i.e. as a rule about all actors at all times, including those of the past. Otherwise, I.)

(48) The deadliest fish in the ocean is Oscar... I (Equivocation on ‘fish’. In the premise, it means a single organism. In the conclusion, it means a type of fish.)

(49) Doctors and lawyers are professionals... V (It is certainly valid if we interpret all the claims as universals, e.g. ‘All professionals and executives are respected. But it would count as invalid if we interpret the claims as generalities, e.g. ‘Professionals and executives are by and large respected.’)

18. Quiz

Take the following quiz to assess how well you understand the definitions. Many of these questions are tricky and require you to think about the definitions carefully.

BEGINNING OF QUIZ

Definition: An argument is sound when (and only when) it is valid and all its premises are true.

(1) If an argument has all true premises,
   a) it is invalid.
   b) it is valid.
   c) it is sound.
   d) none of the above.

(2) You have come across an argument for which there is a possible world where all the premises are true and the conclusion is true. What can you conclude from this?
   a) the argument is valid.
   b) the argument is good.
   c) the argument is sound.
   d) none of the above.

(3) If there is no possibility where all the premises are true, then the argument will be
   a) invalid.
   b) valid and sound.
   c) a contradiction.
   d) valid and unsound.

(4) The conclusion of a sound argument
   a) is true.
   b) might be true or might be false.
   c) is always valid.
   d) is false in the possible worlds where the premises are all true.

True or False

(5) As a rule, if an argument has all false premises, it cannot be valid.
(6) As a rule, if an argument has one false premise, it cannot be valid.
(7) An argument can be valid even if it has a false conclusion.
(8) A valid argument always has a true conclusion whenever the premises are true.
(9) An invalid argument always has a false conclusion whenever the premises are true.
(10) As a rule, if an argument has all false premises and a false conclusion, then it has to be invalid.
(11) As a rule, if an argument has all false premises and a true conclusion, then it has to be invalid.
(12) As a rule, if an argument has all true premises and a false conclusion, then it cannot be valid.
(13) As a rule, if an argument has a false conclusion, then it cannot be valid.
(14) As a rule, if an argument has a contradictory conclusion, then it cannot be valid.
(15) As a rule, if an argument has all false premises and true conclusion, then it cannot be valid.
(16) An invalid argument always has a false conclusion whenever the premises are true.
(17) As a rule, a valid argument always has a false conclusion if one of its premises is false.
(18) A valid argument can have a false conclusion, but only if one of the premises is false.

END OF QUIZ

Also, it may be useful to check whether you can answer the following additional questions.

Why are the following definitions of ‘counterexample’ defective?

- A counterexample is the case that proves the conclusion is false.
- A counterexample is when the premises are true and the conclusion is false.
- A counterexample is a situation that proves that the argument is invalid.
- A counterexample is a situation where the if the premises are true, the conclusion is false.
- A counterexample to an argument is a possible situation where all the premises are true and the conclusion is false.

Why are the following definitions of ‘validity’ defective?

- A valid argument is a case where the premises are true and the conclusion is true as well.
- A valid argument is a case where the premises are true and therefore the conclusion has to be true.
- A valid argument is an argument where if the premises are true, then the conclusion is true.
- A valid argument is an argument where the premises are true and the conclusion can’t be false.
- An argument is valid only when it has no counterexamples.
- An argument is valid if and only if it is not invalid.

19. Argument Forms

Large classes of arguments share the same features. We can exploit this by using argument forms.

By dealing with the form of the argument we can deal with all the arguments that share that form. This will permit us to take a shortcut. Sometimes we won’t have to stretch our imaginations thinking of possible counterexamples. Instead we can just look at the overall structure of the argument to tell whether it is valid.

**Exercise 4.** *Which of the following arguments are valid?*

- Brad went to school, and Jill stayed home sick. 
  Thus, Brad went to school.
- Horses usually weigh over 800 lbs. and they have to eat a lot to maintain their weight. 
  Thus, Horses usually weigh over 800 lbs.
- The Tralfamadorians offered an ulkin to the Earthlings, and then took one of the Earthlings away for experiment. 
  Thus, the Tralfamadorians offered an ulkin to the Earthlings.

Any argument of the form

\[\alpha \text{ and } \beta.\]

Thus, \(\alpha\).

is a valid argument.

**An argument form** is a generic type of argument, a kind of argument skeleton. It usually contains some English words and some Greek letters.

Here is an argument form:

\[\text{Because } \delta, \text{ it isn’t so that } \gamma.\]

\[\text{Thus, } \gamma.\]

Here is another:

\[\text{It is crazy to believe that } \beta.\]

\[\text{It is possible that } \epsilon.\]

\[\text{Thus, it is likely that } \epsilon, \text{ but } \alpha.\]

When we fill in the Greek letters with statements, we get an argument.

Here we can fill in consistently to make an argument:

Let \(\delta = \text{“Everybody was able to see the screen.”}\)

Let \(\gamma = \text{“Yolanda couldn’t sit up straight.”}\)

\[\text{Because Yolanda couldn’t sit up straight, it isn’t so that everybody was able to see the screen.}\]

\[\text{Thus, everybody was able to see the screen.}\]

Here is our other argument form filled in to make an argument:

Let \(\beta = \text{“NASCAR is fun to watch.”}\)

Let \(\epsilon = \text{“Spinach is my favorite food.”}\)

Let \(\alpha = \text{“I think I’m going to become a druid.”}\)

\[\text{It is crazy to believe that NASCAR is fun to watch.}\]

\[\text{It is possible that spinach is my favorite food.}\]

\[\text{Thus, it is likely that NASCAR is fun to watch, but I think I’m going to become a druid.}\]
19.1. Fine Points on Argument Forms.

- Argument forms are constructed so that no matter what statements you plug in, the resulting sentences are grammatical and meaningful.
- Don’t try to preserve the capitalization of the statements being substituted. Just make the finished argument have the correct capitalization.
- You have to be consistent when plugging in statements. If the same Greek letter appears in multiple parts of the argument form, you have to plug a single statement in for all instances of that Greek letter.
- An argument form when fully filled in will always yield an argument, but in general, the argument can be either valid or invalid.

Here is another argument form:

\[
\begin{align*}
\text{Unless } & \alpha, \beta. \\
\text{Thus, } & \alpha \text{ and } \beta.
\end{align*}
\]

Is the following argument an instance of this argument form?

\[
\begin{align*}
\text{Unless I feel like it, I'm not going swimming.} \\
\text{Thus, I'm not going to the park, and I'm not going swimming.}
\end{align*}
\]

Here is the same argument form:

\[
\begin{align*}
\text{Unless } & \alpha, \beta. \\
\text{Thus, } & \alpha \text{ and } \beta.
\end{align*}
\]

The following is an instance of this argument form:

\[
\begin{align*}
\text{Unless I feel like it, I'm not going swimming.} \\
\text{Thus, I feel like it, and I'm not going swimming.}
\end{align*}
\]

Let \(\alpha = \"I feel like it.\"

Let \(\beta = \"I'm not going swimming.\"

Here is another argument form:

\[
\begin{align*}
\alpha \text{ and } & \beta. \\
\delta. \\
\text{Thus, } & \alpha \text{ and } \beta.
\end{align*}
\]

Is the following assignment of variables allowed?

Let \(\alpha = \"Jodi thinks I'm an idiot.\"

Let \(\beta = \"Trent thinks Jodi is right.\"

Let \(\delta = \"Jodi thinks I'm an idiot.\"

\[
\begin{align*}
\text{Jodi thinks I'm an idiot, and Trent thinks Jodi is right.} \\
\text{Jodi thinks I'm an idiot.} \\
\text{Thus, Jodi thinks I'm an idiot, and Jodi thinks I'm an idiot.}
\end{align*}
\]

19.2. Valid Argument Forms. The reason we care about argument forms, is that there are some argument forms that are special. These special argument forms are constructed in such a way that no matter what statements are substituted in for the Greek letters, the resulting argument is valid. These special argument forms are called valid argument forms.

Every instance of a valid argument form is a valid argument.

Here are some valid argument forms:

\[
\begin{align*}
\alpha \text{ but } & \beta. \\
\text{Thus, } & \beta.
\end{align*}
\]
19.3. **Invalid Argument Forms.** An invalid argument form is a form that isn’t valid. That means an invalid argument form has at least one instance that is an invalid argument.

For valid argument forms, all argument instances are valid.

For invalid argument forms, at least one argument instance is not valid.

Here is an invalid argument form:

\[
\begin{align*}
\alpha. \\
\beta. \\
\text{Thus, } \alpha \text{ and } \beta. \\
\alpha \text{ or } \beta. \\
\text{It is not the case that } \alpha. \\
\text{Thus, } \beta. \\
\text{If } \alpha, \text{ then } \beta. \\
\alpha. \\
\text{Thus, } \beta. \\
\alpha \text{ unless } \beta. \\
\text{It is not the case that } \beta. \\
\text{Thus, } \alpha.
\end{align*}
\]

Here is an invalid instance of this argument form. This argument proves that the form is invalid:

Let \( \alpha = \) “Bill is a good snooker player.”
Let \( \beta = \) “Rust forms on old engine parts.”
Let \( \delta = \) There is gold in Alaska.”

\[
\begin{align*}
\text{Bill is a good snooker player.} \\
\text{Rust forms on old engine parts.} \\
\text{Thus, there is gold in Alaska.}
\end{align*}
\]

Here is the same invalid argument form:

\[
\begin{align*}
\alpha. \\
\beta. \\
\text{Thus, } \delta.
\end{align*}
\]

Here is a valid instance of this argument form. Having a valid argument as an instance does not mean the argument form is valid!

Let \( \alpha = \) “Roses usually bloom in May.”
Let \( \beta = \) “Fireflies appear in June.”
Let \( \delta = \) “Roses usually bloom in May, and fireflies appear in June.”

\[
\begin{align*}
\text{Roses usually bloom in May.} \\
\text{Fireflies appear in June.} \\
\text{Thus, roses usually bloom in May, and fireflies appear in June.}
\end{align*}
\]

19.4. **Summary on Argument Forms.** To tell whether an argument form is invalid, fish around plugging in any statements you want until you get an invalid argument. If you find an invalid argument, the form is invalid. If you can’t find an invalid argument, the form is valid. Strategy Tip: Just substitute random,
unrelated statements into an argument form. If the argument form is invalid, it will usually reveal itself very quickly.

**Exercise 5.** *Mark the following argument forms with ‘Valid’ or ‘Invalid’.*

α because β.
Thus, β.

α; however, β.
Thus, it isn’t the case that α.
Most people believe that β.
Thus, β.

It isn’t the case that α.
It isn’t the case that β.
Thus, neither α nor β.

Rick is extremely confident that δ.
Thus, Rick believes that δ.

Either β or α.

β.
Thus, α.

β unless δ.
Thus, β.

α.
β.
δ.
Thus, α, β, and δ.
 α or β.
δ.
Thus, δ and β.

True or False Quiz

(1) As a rule, if an argument is invalid, then it is an instance of some invalid argument form.

(2) As a rule, if an argument is invalid, then it is an instance of some valid argument form.

(3) As a rule, if an argument is valid, then it is an instance of some invalid argument form.

(4) As a rule, if an argument is valid, then it is an instance of some valid argument form.
Propositional Logic

In this chapter, we will construct a formal logic called ‘Propositional Logic’. That means we will define a formal language with rules for grammatically correct statements in the formal language, and techniques for determining the validity of arguments, consistency of sentences, etc., expressed in the artificial language.

Propositional logic is designed to capture more or less the underlying logic of various sentential connectives, i.e. combinatorial relations among parts of sentences that can be interpreted as propositional.

1. The Formal Language of Propositional Logic

We are going to develop a logic that can deal with the four following logical connections between sentences:

- Conjunction: ‘and’
- Disjunction: ‘or’
- Negation: ‘not’
- Conditional: ‘If... then...’

The good thing about propositional logic is that it simplifies arguments for us. The argument

$\text{The cat is in the hat, and the dog is in the fog.}$
$\text{Thus, the cat is in the hat.}$

gets translated into the argument

$C \& D$
$\quad C$

where $C$ is “The cat is in the hat,” and $D$ is “The dog is in the fog.”

The benefit is that after breaking down complicated arguments into their component sentences (or clauses) and a handful of logical connectives like ‘and’, we’ll be able to do some simple manipulations that will tell us with certainty whether the argument is valid. We won’t have to worry about overlooking potential counterexamples.

1.1. Limitations of Propositional Logic. Because the logic only deals with the kinds of logical relationships that exist between simple sentences and compound sentences, there are many aspects of logic that the logic mishandles.

The following arguments are valid, but will appear to be invalid if we use propositional logic because none of the sentences are compound sentences using ‘and’ or ‘or’ or ‘not’ or ‘if...then...’.

$\text{The gong was heard making a loud sound.}$
$\text{Thus, the gong was audible.}$
Jane is a grandmother.
Thus, Jane at some point had a child.
Wendy is a bank teller.
All bank tellers work at some or other bank.
Thus, Wendy works at a bank.

The way to translate into propositional logic is to rearrange the premises and conclusion so that they look like sentences (clauses) connected with the following expressions:

‘… and…’
‘it is not the case that…’
‘… or…’
‘if… then…’

2. Conjunction

Conjunction is just the fancy name for ‘and’. To translate sentences that use the word ‘and’, we first rewrite them so that they have stand-alone clauses connected with the word ‘and’.

Haley and Rachel have a birthday next week.
**Haley has a birthday next week, and Rachel has a birthday next week.**

Kendrick bought some chocolate bars and some panty hose today.
**Kendrick bought some chocolate bars today, and Kendrick bought some panty hose today.**

Victoria is going to register her car, inflate her tires, and change her oil.
**Victoria is going to register Victoria’s car, and Victoria is going to inflate Victoria’s tires, and Victoria is going to change Victoria’s oil.**

The refund check and signed receipt must be given to the administrator in charge and the secretary at the front desk, respectively.
**The refund check must be given to the administrator in charge and the signed receipt must be given to the secretary at the front desk.**

You can fool some of the people all of the time, and you can fool all of the people some of the time, but you can’t fool all of the people all of the time.
**You can fool some of the people all of the time, and you can fool all of the people some of the time, and you can’t fool all of the people all of the time.**

Larry, Moe, and Curly are stooges.
**Larry is a stooge, and Moe is a stooge, and Curly is a stooge.**
Larry, Moe, and Curly are The Three Stooges.
Wrong: Larry is The Three Stooges, and Moe is The Three Stooges, and Curly is The Three Stooges.
Right: Larry, Moe, and Curly are The Three Stooges.
Right: Larry is one of The Three Stooges, and Moe is one of The Three Stooges, and Curly is one of The Three Stooges.

Julie visited an ancient and expansive home yesterday.
Julie visited an ancient and expansive home yesterday.
Julie visited an ancient home yesterday, and Julie visited an expansive home yesterday.

These are conjunctions:
Seth stayed home, but Liz went out.
Seth stayed home, yet Liz went out.
Seth stayed home; however, Liz went out.
Seth stayed home, even though Liz went out.

This is not a conjunction:
Seth stayed home because Liz went out.

The difference is this:
If “Seth stayed home,” is true and that “Liz went out,” is true, then it is certain that “Seth stayed home, but Liz went out,” is also true. However, we cannot tell just from the fact that “Seth stayed home,” is true and that “Liz went out,” is true, whether “Seth stayed home because Liz went out,” is true.

2.1. Truth-functional Connectives. For connectives like ‘but’, ‘yet’, ‘and’, and ‘even though’, knowing the truth and falsity of the connected clauses is enough to tell us what the truth value of the conjunction is.

<table>
<thead>
<tr>
<th>Seth stayed home.</th>
<th>Liz went out.</th>
<th>Seth stayed home, but Liz went out.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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<td>T</td>
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<td>T</td>
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<td>F</td>
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<td>F</td>
</tr>
</tbody>
</table>

2.2. Non-truth-functional Connectives. For connectives like ‘because’, knowing the truth and falsity of the connected clauses is NOT enough to tell us what the truth value of the conjunction is.

<table>
<thead>
<tr>
<th>Seth stayed home.</th>
<th>Liz went out.</th>
<th>Seth stayed home because Liz went out.</th>
</tr>
</thead>
<tbody>
<tr>
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<td>?</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
2.3. ‘But’ vs. ‘And’. There is no logical difference between ‘but’ and ‘and’:
The truth table for them is the same. The difference in their meaning is in the
conversational implications they typically possess. ‘But’ usually implies some kind
of contrast from what came before.

“Seth stayed home, but Liz went out,” implies all of the following statements:
1. Seth stayed home.
2. Liz went out.
3. Liz’s going out is remarkable or surprising given that Seth’s stayed home.

The inference to the above propositions 1 and 2 are logically valid because
1. You can’t consistently say “Seth stayed home, but Liz went out, and Seth didn’t
   stay home.”
2. You can’t consistently say “Seth stayed home, but Liz went out, and Liz didn’t
   go out.”
3. But, you can consistently say “Seth stayed home, but Liz went out, and it is not
   at all surprising or remarkable that Liz went out.”

3. Negation

Negation is the expression of a negative. Negatives appear commonly with the
word ‘not’ and with negative prefixes like ‘in-’ and ‘un-’. To translate negative
sentences, we first rewrite them so that they start off with the expression “It is not
the case that”.

\[
\begin{align*}
\textit{Marvin is not sick.} \\
\text{It is not the case that Marvin is sick.} \\
\textit{Dexter was indispensable to his team.} \\
\text{It is not the case that Dexter was dispensable to Dexter’s team.}
\end{align*}
\]

This nail has never been struck with a hammer.
It is not the case that this nail has (ever) been struck with a hammer.
Let \( H = \text{“This nail has been struck with a hammer.”} \)
\( \neg H \)

4. Subtleties with Negation

Three common cases where ‘not’ cannot be pulled to the front:
Intensional Contexts (belief, speech, desire)
Quantifiers (all, some, none)
Possibility/Necessity (must, might, may, should, could)
Cases where ‘un-’, ‘in-’, ‘im-’ cannot be pulled to the front: Contraries vs.
Negations

4.1. Intensional Contexts. In some cases, the sentence can be negative in
a way that doesn’t allow you to rephrase it using “It is not the case that.”
Wendy said that she did not have herpes.
Wrong: It is not the case that Wendy said she did have herpes.
Right: \textit{Wendy said that she did not have herpes.}
It is written on that sign that smoking is not permitted.
Wrong: It is not the case that it is written on that sign that smoking is permitted.
Right: It is written on that sign that smoking is not permitted.

Negatives that occur in the context of speech, writing, belief, thought, conjecture, etc. cannot usually be translated as a negation. These contexts are called intensional contexts.

Whenever you have ‘believes that’, ‘thinks that’, ‘said that’, ‘wrote that’, ‘wished that’, you often won’t be able to extract the logic from what follows. If the rephrased sentence doesn’t mean the same thing as the original, you should just treat the entire belief statement (or speech statement) as a simple sentence.

Usually, you will be able to extract conjunctions, but usually not negations, and usually not disjunctions.

Good Translation:

The official thought that the Kenyan and the Norwegian had finished the race.

The official thought that the Kenyan had finished the race, and the official thought that the Norwegian had finished the race.

Bad Translation:

Tim instructed his mother not to go into his room.

It is not the case that Tim instructed his mother to go into his room.

Questionable Translation:

The kid wants either cherry or strawberry. (He just wants one of the red flavors, and he doesn’t care which.)

Either the kid wants cherry or the kid wants strawberry. (He wanted a specific flavor, but I forget which one. All I can remember is that it is a red flavor.)

Good Translation:

He didn’t believe that he wasn’t going to make it home.

It is not the case that he believed that he wasn’t going to make it home.

Good Translation:

Alice didn’t say she was in trouble.

It is not the case that Alice said she was in trouble.

Is the next one a good translation?

I don’t know whether we’ll have a blizzard this week.

It is not the case that I know whether we’ll have a blizzard this week.

Is the next one a good translation?

I don’t think Jill is creative.

It is not the case that I think Jill is creative.

Remember: Statements are logically equivalent if they are true in all the same circumstances (and false in all the same circumstances).
4.2. Quantifiers. Whenever you have words like ‘everyone’, ‘all’, ‘someone’, ‘some’, you often won’t be able to extract the logic from what follows. If the rephrased sentence doesn’t mean the same thing as the original, you should just treat the entire statement as a simple sentence.

Usually, you will be able to extract conjunctions, but usually not negations, and usually not disjunctions.

Good Translation:
- Everyone is wearing a cheap tie-dye T-shirt and beret.
- Everyone is wearing a cheap tie-dye T-shirt, and everyone is wearing a beret.

Bad Translation:
- Some people don’t take a shower everyday.
- It is not the case that some people take showers everyday.

Bad Translation:
- Everyone is either male or female.
- Either everyone is male or everyone is female.

4.3. Negative Quantifiers. ‘no one’ = ‘not someone’ = ‘not one person’ = ‘not one or more people’
- nobody’ = ‘not somebody’ = ‘not one person’ = ‘not one or more people’
- ‘nothing’ = ‘not something’ = ‘not one thing’ = ‘not one or more things’
- ‘nowhere’ = ‘not somewhere’ = ‘not one place’ = ‘not one or more places’

No one caught you smoking dope.
- It is not the case that someone caught you smoking dope.
- Nobody wants to be your friend.
- It is not the case that somebody wants to be your friend.
- Nothing is forgiven in this household.
- It is not the case that something is forgiven in this household.
- It is not the case that one or more things are forgiven in this household.
- The contraband is nowhere to be found.
- It is not the case that the contraband is somewhere to be found.

4.4. Possibility/Necessity. Whenever you have words like ‘might’, ‘must’, ‘may’, ‘possible’, ‘necessary’, ‘need’, ‘should’, ‘ought’, you often won’t be able to extract the logic from what follows. If the rephrased sentence doesn’t mean the same thing as the original, you should just treat the statement as a simple sentence.

Good Translation:
- Cliff must own a bike and a helmet.
- Cliff must own a bike, and Cliff must own a helmet.

Bad Translation:
- Cliff might own a bike and a helmet.
- Cliff might own a bike, and Cliff might own a helmet.

Bad Translation:
Amy must not go to Vegas.

**It is not the case that Amy must go to Vegas.**

Bad Translation:

It’s possible that you don’t have chlamydia.

**It is not the case that it is possible that you have chlamydia.**

4.5. Contraries vs. Negations. Two properties are contrary if they cannot be possessed by the same object at the same time.

Two properties are negatives of one another if failure to have the one property implies that you have the other.

Example of contraries: red and blue

Example of negations: colored and colorless

Which translations are good?

Joann is unhappy.

**It is not the case that Joann is happy.**

Reggie was unfazed by the insult.

**It is not the case that Reggie was fazed by the insult.**

We stumbled upon some untoward consequences.

**It is not the case that we stumbled upon some toward consequences.**

Antonio’s shoelace is unraveled.

**It is not the case that Antonio’s shoelace is raveled.**

Cindy didn’t invite Nathan, and we didn’t press the issue with her.

Correct: It is not the case that Cindy invited Nathan, and it is not the case that we pressed the issue with Cindy.

N = “Cindy invited Nathan.”

I = “We pressed the issue with Cindy.”

\[ \neg N \land \neg I \]


Wrong: It is not the case that chameleons eat birds, and chameleons only eat insects and chameleons only eat fruit.

Right: It is not the case that chameleons eat birds, and chameleons only eat insects and fruit. B = “Chameleons eat birds.”

E = “Chameleons only eat insects and fruit.”

\[ \neg B \land E \]

Both Bill and Ted did not respond.

Wrong: It is not the case that (Bill responded and Ted responded).

Right: It is not the case that Bill responded, and it is not the case that Ted responded.

B = “Bill responded.”

T = “Ted responded.”

\[ \neg B \land \neg T \]

Bill and Ted did both not respond.

Wrong: It is not the case that Bill responded, and it is not the case that Ted responded.

Right: It is not the case that both (Bill responded and Ted responded).
B = “Bill responded.”
T = “Ted responded.”
¬(B&¬T)

Summary of technique:
Step 1: Try to pull the negations (not, un-, im-) out in front of the sentence by making them “It is not the case that”
Step 2a: Hard but always works in theory:
Check the meaning of the two sentences. Are they true in all the same circumstances?
Step 2b: Easy but doesn’t always work.
Make sure you don’t pull the negation through words that express modality (might, must), intentional arity (thought, believed), quotation (said, wrote), or quantity (all, some, none).
Step 3: Anywhere you see “It is not the case that” you put a ¬. Anywhere you see an ‘and’ connecting sentences, put a ‘&’. Anywhere you see words that match a given atomic sentence, put the corresponding letter down.

5. Scope

Consider how to symbolize the following statement:
It is not the case that Trevor left his book behind and Trevor went
to class.
B = “Trevor left Trevor’s book behind.”
C = “Trevor went to class.”

You could translate it in two different ways.
¬B&C
¬(B&C)

So it is unclear whether Trevor went to class.

5.1. Well-formed Formulas. In our formal language, we don’t want any
ambiguity, so we have to make rules that leave the structure of the sentence perfectly clear. We do this by setting up some formal rules for what counts as a well-formed formula (wff):

Any capital letter is a wff.
If α is a wff, then ¬α is a wff.
If α is a wff, and β is a wff, then (α&β) is a wff.
If α is a wff, and β is a wff, then (α ∨ β) is a wff.
If α is a wff, and β is a wff, then (α ⊃ β) is a wff.
Nothing else is a wff.

As a relaxing of the rules, we allow ourselves to drop the outermost parentheses.
Why have this new concept of the wff? (Why not just talk about what statements are legal?) There’s no good reason yet. In propositional logic, a wff is the same thing as a statement, but later, with more complicated logic, they will be different. We are just getting used to the terminology that we will have to use later.

Which of the following are legitimate sentences (wffs)?
Example 4.

\[ (-X \& \neg X) \]
\[ (P \& Q \& R) \]
\[ \neg (E \lor g) \]
\[ (\neg T \supset (T \lor \neg \neg \neg S)) \]
\[ ((A \& B) \& \neg (Y \& R)) \]
\[ (\lor R \& P) \]
\[ (F \& \neg F) \]
\[ (T \& \neg H) \]
\[ ((I \& W) \supset K) \]
\[ \neg \neg P \& Q \]

The scope of a connective is the expression containing the connective, any the wffs it connects, any parentheses that directly enclose the connective, and nothing else.

Examples from Arithmetic:

- The scope of the ‘×’ in the expression \(4 + (3 \times 5)\) is marked in bold blue: \(4 + (3 \times 5)\).
- The scope of the ‘+’ in the expression \(4 + (3 \times 5)\) is marked in bold blue: \(4 + (3 \times 5)\).
- The scope of the ‘-’ in the expression \(-8 + (1 + 2)\) is marked in bold blue: \(-8 + (1 + 2)\).
- The scope of the ‘-’ in the expression \(6 + -(1 + 2)\) is marked in bold blue: \(6 + -(1 + 2)\).

The scope of an arithmetic operator is the operator, the numbers it connects and any parentheses around them.

Scope in propositional logic works much like in arithmetic, but a little easier. To make legitimate wffs, we have to put parentheses around any use of \&, \lor, or \supset. So you can think of the \&, \lor, and \supset like ‘+’ signs and the \neg like the minus sign that indicates a negative number.

The scope of the second \& is marked in bold blue: \((K \& \neg (B \& H))\).
The scope of the \neg is marked in bold blue: \((K \& \neg (B \& H))\).
The scope of the first \& is marked in bold blue: \((K \& \neg (B \& H))\).

Remember: \((A \neg B)\) is not a legitimate sentence. The \neg does not work like a minus sign that signifies subtraction. In arithmetic, \((5-3)\) is shorthand for \((5 + -3)\). In logic, we have to write out sentences like \((A \& \neg B)\). No shorthand. We use a bit of shorthand by optionally omitting the outermost parentheses. But the rules otherwise stay the same.

The scope of the \lor is marked in bold blue: \(\neg P \& (Q \lor \neg P)\). The scope of the first \neg is marked in bold blue: \(\neg P \& (Q \lor \neg P)\). The scope of the second \neg is marked in bold blue: \(\neg P \& (Q \lor \neg P)\). The scope of the \& is marked in bold blue: \(\neg P \& (Q \lor \neg P)\).
\[(A\&B)\&(\neg(C\&D))\]
\[\neg(A\&B)\&(C\&D)\]
\[\neg((A\&B)\&(C\&D))\]
\[\neg\neg(A\&B)\&\neg(C\&D)\]

The **main connective** is the connective whose scope is the entire wff.

Examples:
The main connective in \((R \lor \neg\neg E)\&(\neg(P \supset W))\) is the \&.
The main connective in \(\neg((\neg G \lor T) \supset \neg Q)\) is the first \(\neg\).

What is the main connective in the sentences below?
\[-(S \lor \neg Q)\& P\]
\[L\&(\neg R \supset \neg\neg E)\]
\[-(R \lor \neg W) \supset -(P\&\neg R)\]
\[-(Y \supset \neg(W\&\neg E))\]

Helpful rule: If the outermost parentheses are missing (as they usually are), look at connectives that are outside all parentheses. If there’s a connective that is not a negation, that’s the main connective. Otherwise it is the negation.

### 6. Truth Table for Conjunction

We define the meaning of ‘\&’ by making it match up with our intuitive understanding of the word ‘and’:

<table>
<thead>
<tr>
<th>Seth stayed home.</th>
<th>Liz went out.</th>
<th>Seth stayed home, and Liz went out.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(A&amp;B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

**Table 1. Truth Table for Conjunction**

### 7. Truth Table for Negation

We define the meaning of ‘\(\neg\)’ by making it match up with our intuitive understanding of the expression ‘it is not the case that’:
8. Truth Tables

Step 1: Label one column for each sentence letter in the statement and a column for the statement whose truth one wants to evaluate.

Step 2: For \(n\) sentence letters, make \(2^n\) rows. (For 1 letter make 2 rows; for 2 letters make 4 rows; for 3 letters make 8 rows; for 4 letters make 16 rows; for 5 letters make 32 rows; etc.)

Step 3: Put T’s and F’s in the columns for the sentence letters. For the first column, alternate T and F. For the second column, alternate T T then F F. The last column should always have the top half T and the bottom half F. If not, you made a mistake or you didn’t make the right number of rows.

Optional Step 4: Fill in T’s and F’s underneath all instances of the sentence letters, making it consistent with the columns on the left. For example, under the C there should be a pattern of T’s and F’s matching the column on the left marked C.

Step 5: Fill in T’s and F’s under the sentence connectives like \& and \(\neg\) whenever you have enough information to do so. Use the rules for these connectives (their
truth tables) to let you know what letter to put under them. For example, the magenta letter is a T because \((A \& C)\) is true when \(A\) is true and \(C\) is true. The blue letter is an F because the conjunction \((A \& C)\) is false when \(A\) is false and \(C\) is true.

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>((A &amp; C))</th>
<th>(\neg B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>TTT</td>
<td>FT</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>FFT</td>
<td>FT</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>TTT</td>
<td>TF</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>FFT</td>
<td>TF</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>TFF</td>
<td>FT</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>FFF</td>
<td>FT</td>
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<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>TFF</td>
<td>TF</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>FFF</td>
<td>TF</td>
</tr>
</tbody>
</table>

To figure out what letter goes under the \&, you need to look at the what the \& connects: \((A \& C)\) and \(\neg B\). Then look under the column of the main connective for each of them. The main connective of \((A \& C)\) is \& and the main connective of \(\neg B\) is \(\neg\). So we take the T under the \& and the F under the \(\neg\) and we use the truth table for conjunction to tell us that a T and an F conjoined together gives F for an answer. Thus, the green letter is F.

Step 6: After every connective has a full column of T’s and F’s, circle the connective with the widest scope. This is column is identified here by being in boldface blue.

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(C)</th>
<th>((A &amp; C))</th>
<th>(\neg B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>TTT</td>
<td>FT</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>FFT</td>
<td>FT</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>TTT</td>
<td>TF</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>FFT</td>
<td>TF</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>TFF</td>
<td>FT</td>
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<td>F</td>
<td>FFF</td>
<td>FT</td>
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<td>TFF</td>
<td>TF</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>FFF</td>
<td>TF</td>
</tr>
</tbody>
</table>

Truth tables tell us what statements are tautologies, what statements are contradictions and what statements are contingent.

We look at the boldface blue column.
• If we see all T’s, the statement is a tautology.
• If we see all F’s, the statement is a contradiction.
• If we see some T’s and some F’s, the statement is contingent.

**Exercise 6.** Prove that the following is a contradiction: “Lithium chlorate is an acid, and it’s not an acid.”
First, we translate it into “Lithium chlorate is an acid, and it’s not the case that lithium chlorate is an acid.”
Let \( A = \) “Lithium chlorate is an acid.”
Then, we translate into symbols: \((A \& \neg A)\)
Then, we do the truth table. The column under the main connective is all F’s, so it’s a contradiction.

<table>
<thead>
<tr>
<th></th>
<th>(A)</th>
<th>(A &amp; \neg A)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

**Exercise 7.** Prove that the following statements are equivalent:
“Keith went to the movies, and Chuck did too.”
“Chuck went to the movies, and Keith did as well.”
First, we translate them into
“Keith went to the movies, and Chuck went to the movies.”
“Chuck went to the movies, and Keith went to the movies.”
Let \( K = \) “Keith went to the movies.”
Let \( C = \) “Chuck went to the movies.”
Then we translate into symbols: \((K \& C)\) and \((C \& K)\).
Then, we do the truth table. Because both columns under the main connective are the same in every row, they are equivalent.

<table>
<thead>
<tr>
<th></th>
<th>(K)</th>
<th>(C)</th>
<th>(K &amp; C)</th>
<th>(C &amp; K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
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<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

**Exercise 8.** Prove that the following statements are equivalent:
“It’s untrue that Amy didn’t pass out.”
“Amy passed out.”
First, we translate into
“It’s not the case that it’s not the case that Amy passed out.”
“Amy passed out.”
Let \( A = \) “Amy passed out.”
Then we translate into symbols: \( \neg \neg A \)
Then, we do the truth table. Because both columns under the main connective are the same in every row, they are equivalent.
2. PROPOSITIONAL LOGIC

Exercise 9. Problem: Show that the two ways of translating “Inky, Blinky, and Pinky are ghosts,” are equivalent.

Let \( I = \) “Inky is a ghost.”
Let \( B = \) “Blinky is a ghost.”
Let \( P = \) “Pinky is a ghost.”

We can translate it as \((I\&B)\&P\) or as \((I\&B\&P)\). This means we don’t have to worry about how to place the parentheses in translating sentences with 2 conjunctions. They are logically equivalent.

Exercise 10. Show that the two sentences below mean different things:
“Heather and Stephanie did not both go to the bar.”
“Heather and Stephanie both did not go to the bar.”
\( H = \) “Heather went to the bar.”
\( S = \) “Stephanie went to the bar.”
\( \neg(H\&S) \)
\((\neg H\&\neg S)\)

They have two rows (the blue ones) where the sentences differ in truth value. Thus, they are inequivalent.

Exercise 11. Evaluate the following argument for validity.
\( \neg(A\&\neg B) \)
\( \neg(B\&A) \)
\((A\&\neg B)\)

The truth table shows (in blue) that we have a counterexample when \( A \) is false and \( B \) is true and we have another counterexample when \( A \) and \( B \) are both false. Thus, the argument is invalid.
Example argument:

*It turned out to be untrue that Eve, Regina and Donna all did their own homework.*
*Jason falsely claimed, “Donna did her own homework, but Regina did someone else’s instead.”*
*Thus, Eve didn’t do her homework.*

We translate this into:

*It is not the case that (Eve did Eve’s homework and Regina did Regina’s homework and Donna did Donna’s homework).*
*It is not the case that (Donna did Donna’s homework, and it is not the case that Regina did Regina’s homework).*
*Thus, it is not the case that Eve did Eve’s homework.*

Let $D = “Donna did Donna’s homework.”$
Let $R = “Regina did Regina’s homework.”$
Let $E = “Eve did Eve’s homework.”$

\[
\neg((E \& R) \& D) \\
\neg(D \& \neg R) \\
D \\
\neg E
\]

The proof is in the truth table. There are no counterexamples. Thus, the argument is valid.

9. Translating Negations with Conjunctions

If the ‘not’ is on the left of the ‘both’, put the ‘¬’ on the left of the conjunction.
If the ‘not’ is on the right of the ‘both’, make a conjunction of negations.

Dave and Carol are not both going.  \(\neg(D \& C)\)
Dave and Carol are both not going.  \((\neg D \& \neg C)\)
Dave and Carol both are not going. \((\neg D \& \neg C)\)
Both Dave and Carol are not going. \((\neg D \& \neg C)\)

10. Disjunction

Disjunction is the fancy name for ‘or’. It is usually expressed with ‘or’ or with “either or” Just like with conjunction, you should express disjunctions by breaking them up into statements connected with ‘or’. Disjunction is symbolized with the wedge: ‘\(\lor\)’.

Dave has cable or a satellite dish.
Dave has cable, or Dave has a satellite dish.
Either Trent or Evan won the tournament.
Trent won the tournament or Evan won the tournament.
You should be able to understand what I’m saying, or you’re an idiot.
Let \(U\) = “You should be able to understand what I am saying.”
Let \(I\) = “You are an idiot.”
\(U \lor I\)

Just as with negation you have to watch out for quantifiers, psychological verbs, and for sentences that express possibility or necessity.

Bad Translation:
Every whole number is either odd or even.
Every whole number is even, or every whole number is odd.

Questionable Translation:
People have to mark the form with either ‘male’ or ‘female’.
People have to mark the form with ‘male’, or people have to mark the form with ‘female’.

Questionable Translation:
The kid wants either cherry or strawberry. (He just wants one of the red flavors, and he doesn’t care which.)
Either the kid wants cherry or the kid wants strawberry. (He wanted a specific flavor, but I forget which one. All I can remember is that it is a red flavor.)

10.1. Truth Table for Disjunction. This is the truth table for \(\lor\):

<table>
<thead>
<tr>
<th>(A)</th>
<th>(B)</th>
<th>(A \lor B)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
<tr>
<td>(F)</td>
<td>(T)</td>
<td>(T)</td>
</tr>
<tr>
<td>(T)</td>
<td>(F)</td>
<td>(T)</td>
</tr>
<tr>
<td>(F)</td>
<td>(F)</td>
<td>(F)</td>
</tr>
</tbody>
</table>

Table 3. Truth Table for Disjunction

The last three lines are totally uncontroversial, but the top line has at least has generated some debate over whether ‘\(\lor\)’ accurately captures the logic of the English ‘or’.

The inclusive sense of ‘or’ is the interpretation where we interpret “\(A\) or \(B\)” as “Either \(A\) or \(B\) or both.” The exclusive sense of ‘or’ is the interpretation where we interpret “\(A\) or \(B\)” as “Either \(A\) or \(B\), but not both.”.
Here are three arguments why the English ‘or’ is inclusive (i.e., given by the truth table for ‘∨’).

Argument I: Suppose you are coming back from the party where you saw many friends including Julie and Elaine. Dirk stops you outside on the street and asks you the following: “I’m looking for my roommate, and I need to find someone who has seen him lately. Have you seen either Julie or Elaine tonight?” What is the most honest response you can give, ‘Yes’ or ‘No’? The answer is ‘Yes.’ That means you agree with the statement, “I have seen either Julie or Elaine tonight.” That means you think the statement, “I have seen Julie tonight, or I have seen Elaine tonight,” is true.

Argument II: Suppose someone says, “Either she had heart surgery, or she had brain surgery.” That seems to imply she didn’t have both surgeries. We can test this out by trying to cancel the inference. We consider, “Either she had heart surgery, or she had brain surgery, and I think she had both surgeries.” If the inference were logical, this sentence would sound contradictory, but it doesn’t. Thus, the inference is conversational, not logical.

Argument III: Suppose Jeff plans to go to the Metallica concert and to sleep there, but he says, “I’ll either go to the Metallica concert tonight, or I’ll catch up on some sleep.” Jeff is probably being misleading because it is improper conversational etiquette to say what is less informative when you are in a position to be more informative.

It is similar to when Jeff tells his girlfriend Lisa that he is going over to Dave’s house, when in fact he is planning to go over to Dave’s house for a token appearance before going next door to pick up Jennifer for their date.

11. Examples using Disjunction

Problem: Show that \((A \lor \neg A)\) is a tautology.

<table>
<thead>
<tr>
<th>A</th>
<th>A \lor \neg A</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Problem: Show \((K \lor C)\) is logically equivalent to \((C \lor K)\).

<table>
<thead>
<tr>
<th>K</th>
<th>C</th>
<th>K \lor C</th>
<th>C \lor K</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
Problem: Prove that the following argument is valid.

_Ethan is going to both Bermuda and Oslo during his vacation._
_During his vacation, Ethan is either going to Bermuda or to Oslo._

B = “Ethan is going to Bermuda during his vacation.”
O = “Ethan is going to Oslo during his vacation.”

<table>
<thead>
<tr>
<th>B</th>
<th>O</th>
<th>B&amp;O</th>
<th>B ∨ O</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>FT</td>
<td>TTF</td>
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<tr>
<td>T</td>
<td>F</td>
<td>TFF</td>
<td>FFT</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>FFF</td>
<td>FFF</td>
</tr>
</tbody>
</table>

There are no rows where the B&O is true and the B ∨ O is false. Thus, the argument is valid.

Problem: Show that \( ((P ∨ R) ∨ S) \) and \( (P ∨ (R ∨ S)) \) are logically equivalent.

<table>
<thead>
<tr>
<th>P</th>
<th>R</th>
<th>S</th>
<th>((P ∨ R) ∨ S)</th>
<th>(P ∨ (R ∨ S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T T T T</td>
<td>T T T T</td>
</tr>
<tr>
<td>F</td>
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<td>T</td>
<td>F</td>
<td>T</td>
<td>T F T T</td>
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<td>F</td>
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<td>F F T T</td>
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<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F F F F</td>
<td>F F F F</td>
</tr>
</tbody>
</table>

Problem: Show that \( ((P&R) ∨ S) \) and \( (P&(R ∨ S)) \) are logically inequivalent.

<table>
<thead>
<tr>
<th>P</th>
<th>R</th>
<th>S</th>
<th>((P&amp;R) ∨ S)</th>
<th>(P&amp;(R ∨ S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>TTT T</td>
<td>TT T T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>FFT T</td>
<td>FT T T</td>
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<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>TFF T</td>
<td>TT F T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>FFF T</td>
<td>FF F T</td>
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<td>TTT F</td>
<td>TT T F</td>
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<td>F</td>
<td>TFF F</td>
<td>TF F F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>FFF F</td>
<td>FF F F</td>
</tr>
</tbody>
</table>

Problem: Show that \( ¬(X&Y) \) and \( (¬X ∨ ¬Y) \) are logically equivalent.

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>¬(X&amp;Y)</th>
<th>¬(X ∨ ¬Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>FT T T</td>
<td>FFFFT</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>TFT T</td>
<td>TFTFT</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>TFT F</td>
<td>TTTTF</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>TFFT</td>
<td>TFTTF</td>
</tr>
</tbody>
</table>

Problem: Show that \( ¬(X ∨ Y) \) and \( (¬X&¬Y) \) are logically equivalent.
11. EXAMPLES USING DISJUNCTION 43

<table>
<thead>
<tr>
<th>X</th>
<th>Y</th>
<th>~(X ∨ Y)</th>
<th>(~X &amp; ~Y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>FTTT</td>
<td>FTFFT</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>FTTT</td>
<td>TFFFF</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>FTTF</td>
<td>FTFTF</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>TFFF</td>
<td>TFFFF</td>
</tr>
</tbody>
</table>

11.1. Translating Disjunctions. If the ‘not’ (possibly from the ‘n’ in ‘neither’) is on the left of the ‘either’, put the ‘¬’ on the left of the disjunction. ¬(D ∨ C)
If the ‘not’ is on the right of the ‘either’, make a disjunction of negations, (¬D ∨ ¬C), or a disjunction with only one : if there is an extra verb to make clear where the disjunction should not go.
Either Dave or Carol is going. Neither Dave nor Carol are going. Either Dave or Carol is not going. Either Dave is or Carol is not going.
Kelly and Margaret or Rex and Jeff are going to the movies.
Kelly and Margaret or Rex are going to the movies. Ambiguous, but probably
Kelly and either Margaret or Rex are going to the movies.
Either Kelly or Margaret and Rex are going to the movies. Still ambiguous, but probably
Christine or Shannon is not going to be in the play.
Either Christine and Shannon are not or Taylor is going to be in the play.
Either Christine or Shannon are not and Taylor is going to be in the play.
Neither Christine nor Shannon are not going to be in the play.
Neither Christine nor Shannon nor Taylor are going to be in the play.
Notice how there are two devices for disambiguating the scope of sentences that involve disjunctions and conjunctions. The first is the use of shortening the conjunction or disjunction of independent clauses into a conjunction or disjunction of the subject.
Let B = “Bill drove the bus.”
Let T = “Ted drove the bus.”
Let W = “We waited.”
“Either Bill and Ted drove the bus, or we waited.”: (B & T) ∨ W
“Either Bill or Ted drove the bus, and we waited.”: (B ∨ T) & W
The second device is to use the “Either...or...” to bracket what counts as the disjunct.
“Either Bill drove the bus, or Ted drove, and we waited.”: B ∨ (T & W)
“Either Bill drove the bus, and we waited, or Ted drove.”: (B & W) ∨ T
Let M = “Melanie slept.”
Let K = “Kara slept.”
Let J = “Jill stayed up.”
“Jill stayed up, and Kara slept, or Melanie slept.” (Ambigious)
“Jill stayed up and Kara or Melanie slept.”: J & (K ∨ M)
“Jill stayed up and either Kara slept or Melanie slept.”: J & (K ∨ M)
“Either Jill stayed up and Kara slept or Melanie slept.”: (J & K) ∨ M
The “neither...nor...” construction is easy to translate. It means “not-(either...or...).” “Neither Melanie nor Kara slept.”
“It is not the case that [either Melanie or Kara slept].”
¬(M ∨ K) which is equivalent to ¬M & ¬K.
'Neither' and 'nor' sometimes exist without each other in which case they mean 'also not.'

"Kara didn’t stay up, and neither did Melanie.": \( \neg K \& \neg M \).

"Kara didn’t stay up, nor did Melanie.": \( \neg K \& \neg M \).

There are some times when the English 'or' is really used as a conjunction. Here is one example:

"Ms. Thompson offered us tea or coffee."

"Ms. Thompson offered us tea, and Ms. Thompson offered us coffee."

Let \( T \) = "Ms. Thompson offered us tea."

Let \( C \) = "Ms. Thompson offered us coffee."

\( T \& C \)

### 12. Gricean Implicature and Robustness

Consider the following argument:

\[
\begin{array}{c}
A \\
A \lor B
\end{array}
\]

Problem: Is this argument valid? Yes...

<table>
<thead>
<tr>
<th>( K )</th>
<th>( C )</th>
<th>( K \lor C )</th>
<th>( C \lor K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
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<td>T</td>
<td>F</td>
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<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Consider one of its instances:

**Some frogs are orange.**

**Thus, some frogs are orange or Rodan attacked Tokyo last week.**

Problem: Is this a good argument? In a sense, yes, and in another sense, no. It certainly preserves truth, but it seems to be awkward. We should try to explain why it seems funny to make such an inference.

#### 12.1. Gricean Implicature

One explanation for the awkwardness of the inference from \( A \) to \( (A \lor B) \) is that it violates the Gricean rule of conversation that you should not be less informative when you are in a position to be more informative at little or no cost.

If I believe that Rodan is not a real creature, has not really attacked Tokyo, etc., then the only reason I believe the disjunction is because I believe the first disjunct—that some frogs are orange. Because that is logically stronger than the disjunction, I should just say “Some frogs are orange.” To say the disjunction will wrongly signal to others that I believe the disjunction to be significantly more likely than either disjunct alone.

Consider the following.

“Jim is in Boston this weekend, or at least somewhere in Massachusetts.”

If all we were concerned about was truth, this sentence would violate Gricean principles because it is equivalent (truth-wise) with “Jim is somewhere in Massachusetts this weekend.” But the sentence does certainly convey something more, namely that the speaker has enough confidence in “Jim is in Boston this weekend,” to state that
claim, but the speaker wants to provide a fallback claim for the audience to adopt if they find out Jim is not in Boston this weekend. The speaker has even more confidence that Jim is somewhere in Massachusetts and believes that this claim will hold up even if Jim is not in Boston. We can say Jim wants to communicate that “Jim is somewhere in Massachusetts this weekend” is robust with respect to the falsity of “Jim is in Boston this weekend.”

We often don’t use the following argument form:

\[
F \\
\text{Thus, } F \text{ or } G.
\]

There are two theories that try to explain this: Paul Grice says we often don’t do it because of conversational conventions. If we are in a position to be more informative, we ought to be more informative. Saying ‘F or G’ is generally less informative than saying ‘F’. Frank Jackson says we often don’t do it because the conclusion is often not robust. If you believe ‘F’, there is no reason in general to believe ‘If not F, then G’.

12.2. Robustness. Another explanation for our unwillingness to find the inference from A to (A \lor B) always reasonable is that there are two things we express when we utter an English conditional: truth and robustness.

When I say “A or B,” I am signaling to you that

1. I believe the sentence (A \lor B) is true.
2. I will still believe the sentence (A \lor B) is true if I become convinced that A is false.
3. I will still believe the sentence (A \lor B) is true if I become convinced that B is false.

The first concerns truth. The second and third concern robustness. We say the proper use of ‘or’ requires robustness with respect to the denial of either disjunct.

What is wrong with

\[
A \\
A \lor B
\]

is that although the inference preserves truth, it does not preserve robustness.

The argument

Some frogs are orange.

Thus, some frogs are orange or Rodan attacked Tokyo last week.

does not lead to a conclusion that is robust with respect to the denial of “some frogs are orange.”

Robustness occurs with other argument forms.

\[
\neg E \\
L \\
\neg (E \& L)
\]

is valid but consider the ordinary language instance

You won’t eat that mushroom.

You will live.

Thus, you won’t eat that mushroom and live.

Although valid, the conclusion is misleading in a context where you are still considering whether to eat the mushroom and I am stating it only because I believe
you will not eat it. That’s because we expect a denial of a conjunction to be robust with respect to the truth of each conjunct.

13. Conditionals

As we know from our rules, \((A \supset C)\) is a sentence.

Some terminology: The sentence before the ‘⊃’ is called the antecedent. The sentence after the ‘⊃’ is called the consequent.

**antecedent ⊃ consequent**

The truth table for ‘⊃’ is

<table>
<thead>
<tr>
<th>A</th>
<th>C</th>
<th>A ⊃ C</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 4. Truth Table for the Material Conditional

The conditional is true if either the antecedent is false or the consequent is true. Otherwise, it is false. The truth table for ‘⊃’ is all we need to know to calculate complex expressions using it.

Here is the truth table for \(\neg(P \supset S) \supset (R \& S)\)

<table>
<thead>
<tr>
<th>P</th>
<th>R</th>
<th>S</th>
<th>(\neg(P \supset S) \supset (R &amp; S))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F F T T T T F T T T T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F T T T T F T T F F</td>
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<td>T</td>
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<td>T F F F F F T T T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F F T F T F T F</td>
</tr>
</tbody>
</table>

13.1. **Translations to and from English.** Consider \(J \supset C\) with

\(J = \text{“Jason bought a new skateboard.”}\)

\(C = \text{“Cliff will be impressed.”}\)

Here are different idioms we have in English to express this idea:

“If Jason bought a new skateboard, then Cliff will be impressed.”

“Cliff will be impressed if Jason bought a new skateboard.”

There are some other expressions that also get translated with ‘⊃’:

“The match lit only if there was oxygen in the room.”

\(L = \text{“The match lit.”}\)

\(O = \text{“There was oxygen in the room.”}\)

\(L \supset O\)

“We could hear the cannon blast only if the cannon was fired.”

\(H = \text{“We could hear the cannon blast.”}\)

\(F = \text{“The cannon was fired.”}\)

\(H \supset F\)
“The only way the couch could have caught on fire is if you were smoking in my apartment.”
“The couch was able to catch on fire only if you were smoking in my apartment.”
\( C = \text{"The couch was able to catch on fire."} \)
\( S = \text{"You were smoking in my apartment."} \)
\( C \supset S \)

\( \supset \) means ‘only if’. However, if you try to translate the ‘\( \supset \)’ as ‘only if’ that will sometimes lead to funny translations:
\( K = \text{"Kelly will flip the switch."} \)
\( L = \text{"The light will come on."} \)
How would we translate \( K \supset L \)?:
“Kelly will flip the switch only if the light will come on.”
Awkward but logically correct:
“Kelly will flip the switch only if the light will come on.”
More natural:
“If Kelly will flip the switch, then the light will come on.”
When translating \( \supset \) into English, use whichever formulation sounds most natural.

13.2. Unless. Sentences with ‘unless’ make a claim, and then give an escape clause. If the escape condition is not satisfied, then the claim stands as stated. If the escape condition is satisfied, the claim is effectively retracted.
Suppose I say: “Unless my car is broken, I will pick you up.”
My claim is a promise that I will pick you up. However, I added an escape clause so that if my car is broken, I’m no longer obligated to pick you up.

Case I: My car is not broken. In this case, I am obligated to pick you up.
Case II: My car is broken. In this case, I might pick you up and I might not. It depends on whether I can get alternative transportation. I’m not making any promises one way or the other.

“Unless my car is broken, I will pick you up.” means the same thing as “If my car is not broken, I will pick you up.” So we translate it as follows:
\( C = \text{"Claim = "I will pick you up."} \)
\( E = \text{"Escape clause = "My car is broken."} \)
\( \neg E \supset C \)

How do we translate “We are going to crash unless you get out of the way”?\)
\( C = \text{"We are going to crash."} \)
\( E = \text{"You get out of the way."} \)
\( \neg E \supset C \)

13.3. If and only if. This gets used in philosophy and mathematics. It has the obvious meaning. \( A \) if and only if \( B \) means \( A \) if \( B \), and \( A \) only if \( B \) which means (If \( B \), then \( A \)) and (\( A \) only if \( B \)) which means (\( B \supset A \))\&(\( A \supset B \)). Example:
“A polygon is a triangle if and only if it has three edges.”

A shorthand way of translating ‘if and only if’ is to use a new symbol: \( \equiv \)
The truth table for the biconditional is the following:
You can do a truth table to prove that \( A \equiv C \) is logically equivalent to (\( B \supset A \))\&(\( A \supset B \)).
People sometimes use the abbreviation ‘iff’ for ‘if and only if’.
Also, philosophers often use the phrase ‘just in case’ to mean the same thing as ‘if and only if’. For example, an object is an artifact just in case it is an object crafted by an intentional agent.

13.4. Summary of Translation Rules.

Let \( G = \) “Gina travels.”
Let \( H = \) “Jill travels.”

When a sentence includes ‘if,’ first check whether it appears as part of an ‘only if’ or an ‘even if’ or an ‘if and only if.’ If it does, use the special rules for these idioms:

- “Gina travels only if Jill does,” becomes \( G \supset J \).
- “Only if Jill travels, does Gina travel,” becomes \( G \supset J \).
- “Gina travels even if Jill does,” becomes \( G \).
- “Even if Jill travels, Gina travels,” becomes \( G \).
- “Gina travels if and only if Jill does,” becomes \((G \supset J) \& (J \supset G)\) or simply \( G \equiv J \).

In cases where the ‘if’ is not part of such an idiom, spell out the sentence in ‘if...then...’ form.

- “Jill travels, if Gina does,” becomes \( G \supset J \).
- “If Gina travels, then Jill does,” becomes \( G \supset J \).
- “If Gina travels, so does Jill,” becomes \( G \supset J \).

The cases where conjunctions are involved have their scope disambiguated by looking at where the ‘if’ and ‘then’ are located. Anything between the ‘if’ and the ‘then’ is bracketed together on the left side of the ‘\( \supset \)’ and everything else (up to the end of the conditional) is bracketed on the right side. Also if subjects or direct objects are conjoined, they are not split by the ‘\( \supset \)’.

Let \( K = \) “Kelly travels.”

- “If Gina and Kelly travel, then Jill does,” becomes \((G \& K) \supset J\).
- “If Gina travels, then Kelly and Jill do,” becomes \( G \supset (K \& J) \).
- “If Gina travels, and Kelly travels, then Jill travels,” becomes \((G \& K) \supset J\).
- “If Gina travels, then if Kelly travels, Jill travels,” becomes \((K \supset G) \supset J\).
- “If Gina travels if Kelly does, then Jill travels,” becomes \((K \supset G) \supset J\).

However, people who make this statement almost surely intend a content which is represented better as \((K \& G) \supset J\). These two propositions are not truth-functionally equivalent, and the reason the second is better is that it avoids some of the funny-business that results from treating conditionals as always true when their antecedent is false.

Table 5. Truth Table for the Biconditional
With ‘unless’, we call the clause that immediately follows the ‘unless’ and call it the escape clause. The other clause is called the main clause. Then we put the negation of the escape clause on the left side of ‘⊃’ and the main clause on the right.

- “Unless Gina travels, Jill does,” becomes \( \neg G \supset J \).
- “Unless Gina doesn’t travel, Jill does,” becomes \( \neg \neg G \supset J \).
- “Jill travels unless Gina does,” becomes \( \neg G \supset J \).
- “Unless Gina and Kelly travel, Jill does,” becomes \( \neg (G \& K) \supset J \).
- “Unless Gina travels, Kelly and Jill travel,” becomes \( \neg G \supset (K \& J) \).
- “Unless Gina and Kelly don’t travel, Jill does,” becomes \( \neg (\neg G \& \neg K) \supset J \).

13.5. A Related Remark on Disjunction. Consider the following translation:

\[ R = \text{It rains.} \]
\[ I = \text{We will have the party inside.} \]
\[ P = \text{We will have the party at the park.} \]

“If it rains, we will have the party inside, or if it doesn’t, we will have it at the park.”

Because the word ‘or’ appears in this sentence, it appears to be a disjunction of two conditionals. However, I think the intended content is pretty clearly a conjunction: \( (R \supset I) \& (\neg R \supset P) \). The reason it is phrased as a disjunction, I think, is that the speaker is trying to emphasize the uncertainty. In effect, they are trying to say, “It will either rain or it won’t; if it does, we will have the party inside; if it doesn’t, we will have it at the park.”

There are other cases where ‘or’ is translated as a conjunction. For example, “You can have a biscuit or you can have toast.” This is clearly meant to imply that you can have a biscuit and it also implies that you can have toast. Thus, it cannot be correctly translated as a disjunction. A disjunction would not imply either disjunct. The disjunction here, I think, is just employed in order to impart the kind of conversational implicature that politely indicates that you are not intended to have both.

13.6. The English Indicative Conditional vs. The Material Conditional. The English indicative conditional is the name for sentences of English of the form “If \( \alpha \), then \( \beta \),” or “\( \alpha \) only if \( \beta \),” or “\( \alpha \) unless \( \beta \),” or some similar expression.

The material conditional is the name for sentences of logic of the form ‘\( \alpha \supset \beta \).’

The philosophical question we want to answer is, “How good of a job does our logic do of translating English conditionals?”

The English conditional makes a claim that is only in force under certain specified circumstances (i.e. when the antecedent is true).

Suppose I say, “If the sign is octagonal, then it’s red.” What I am doing is conditionally asserting that the sign is red. There are two ways things can turn out:

1. The sign is octagonal. In this case, I am committed to the statement that the sign is red.
2. The sign is not octagonal. In this case, I am not committed to anything.

Two reasons for thinking that ‘⊃’ does a good job of capturing the meaning of English conditionals: 1. Conditionals that are promises have the same truth table as ‘⊃’. 2. Conditionals can be translated as a disjunction.
2. PROPOSITIONAL LOGIC

<table>
<thead>
<tr>
<th>It’s octagonal.</th>
<th>It’s red.</th>
<th>If it’s an octagon, then it’s red.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
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<td>?</td>
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<td>T</td>
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<td>F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>?</td>
</tr>
</tbody>
</table>

Argument I: Consider “If you get hurt, I will help you.”
Think about the conditional as a promise:
If I promise you something and I keep the promise, then my statement of the promise is true.
If I promise you something and I break the promise, then my statement of the promise is false.

<table>
<thead>
<tr>
<th>You get hurt.</th>
<th>I will help you.</th>
<th>If you get hurt, I will help you.</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T (I kept my promise)</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T (I kept my promise)</td>
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<tr>
<td>T</td>
<td>F</td>
<td>F (I broke my promise)</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T (I kept my promise)</td>
</tr>
</tbody>
</table>

Argument II: Consider the following translations:
“If I left my keys at home, then my roommate can bring them.”
“Either I did not leave my keys at home, or I did leave my keys at home and my roommate can bring them.”
“If you ate the pizza, then the fridge is empty.”
“Either you didn’t eat the pizza, or you did eat the pizza and the fridge is empty.”

P = “You ate the pizza.”
E = “The fridge is empty.”

<table>
<thead>
<tr>
<th>P</th>
<th>E</th>
<th>(\neg P \lor (P \land E))</th>
<th>(P \supset E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T T</td>
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<td>F</td>
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<td>F F</td>
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<td>F</td>
<td>F</td>
<td>T</td>
<td>F T</td>
</tr>
</tbody>
</table>

13.7. Examples of Arguments. Consider the following argument:

If you mow my lawn, I will pay you $20.
If you don’t mow my lawn, I won’t pay you $20.

M = “You mow my lawn.”
P = “I will pay you $20.”

\[M \supset P\]
\[\neg M \supset \neg P\]

Is this valid? No

Consider the following argument in the context of a bet about US history:
If James Madison was a US president, then I will pay you $20.
If I don’t pay you $20, then James Madison was not a US president.
M = “James Madison was a US president.”
P = “I will pay you $20.”
13. CONDITIONALS

<table>
<thead>
<tr>
<th>M</th>
<th>P</th>
<th>M ⊃ P</th>
<th>¬M ⊃ ¬P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T T T</td>
<td>FT FT</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F T T</td>
<td>FT FT</td>
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<tr>
<td>T</td>
<td>F</td>
<td>T F F</td>
<td>FT TF</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F T F</td>
<td>TF TF</td>
</tr>
</tbody>
</table>

\[ M \supset P \]
\[ \neg P \supset \neg M \]
Is this valid? Yes

Consider the following argument:
*If lightning struck, we heard a loud noise.*
*If we heard a loud noise, lightning struck.*

L = “Lightning struck.”
H = “We heard a loud noise.”

\[ L \supset H \]
\[ H \supset L \]
Is this valid? No

<table>
<thead>
<tr>
<th>H</th>
<th>L</th>
<th>L ⊃ H</th>
<th>H ⊃ L</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T T T</td>
<td>T T T</td>
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<tr>
<td>F</td>
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<tr>
<td>F</td>
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<td>F T F</td>
<td>F T F</td>
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</tbody>
</table>

13.8. Argument Forms for Conditionals. There are four simple but important argument forms that use conditionals. Here are their instances.

Modus Ponens:
*There is gold in the mountain.*
*If there is gold in the mountain, I will be rich.*
*Thus, I will be rich.*

which is valid.

<table>
<thead>
<tr>
<th>A</th>
<th>C</th>
<th>A ⊃ C</th>
<th>A</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T T T</td>
<td>T</td>
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<td>F</td>
<td>F</td>
<td>F T F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

Denying the antecedent:
There is no gold in the mountain.
If there is gold in the mountain, I will be rich.
Thus, I will not be rich.

which is invalid.

\[
\begin{array}{c|c|c|c|c}
A & C & A \supset C & \neg A & \neg C \\
T & T & T & T & T \\
F & T & T & T & F \\
T & F & T & F & F \\
F & F & T & F & T \\
\end{array}
\]

Affirming the consequent:
I will be rich.
If there is gold in the mountain, I will be rich.
Thus, there is gold in the mountain.

which is invalid.

\[
\begin{array}{c|c|c|c|c}
A & C & A \supset C & A & C \\
T & T & T & T & T \\
F & T & T & T & F \\
T & F & T & F & F \\
F & F & T & F & F \\
\end{array}
\]

Modus Tollens:
I will not be rich.
If there is gold in the mountain, I will be rich.
Thus, there is no gold in the mountain.

which is valid.

\[
\begin{array}{c|c|c|c|c}
A & C & A \supset C & \neg C & \neg A \\
T & T & T & F & T \\
F & T & T & F & F \\
T & F & T & F & T \\
F & F & T & F & F \\
\end{array}
\]

Why is it tempting to use the invalid argument forms?
Reason I: Often there is a conversational implicature that the conditional is really an ‘if and only if’. Example: “If you don’t brush your teeth, your teeth will rot out.”

Reason II: Sometimes it’s good scientific reasoning to use reasoning that looks like affirming the consequent. The following argument is the kind of argument we would say is normally reasonable in ordinary circumstances.

Joni had marijuana breath, was giggling a lot, and had “the munchies.”
If Joni were stoned on marijuana, she would have marijuana breath, would be giggling a lot, and would have the munchies.
Thus, Joni is probably stoned.
13.9. Paradoxes of the Material Conditional. Is the following statement a tautology, a contradiction, or a contingent statement?

“If a meteor will hit Australia in 2014, then Napoleon’s second favorite color was yellow, or if Napoleon’s second favorite color was yellow, then a meteor will Australia in 2014.”

\((A \supset Y) \lor (Y \supset A)\)

<table>
<thead>
<tr>
<th>A</th>
<th>Y</th>
<th>((A \supset Y) \lor (Y \supset A))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T T T T T</td>
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<tr>
<td>F</td>
<td>T</td>
<td>F T T T F</td>
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<td>T</td>
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<td>T F F T F</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F F F T F</td>
</tr>
</tbody>
</table>

It’s a tautology.

Is the following statement a tautology, a contradiction, or a contingent statement?

“Either (if the Eiffel Tower is in Paris, then the Taj Majal is in Peru), or (if the Eiffel Tower is in Paris, then the Taj Majal is not in Peru.)”

\((E \supset T) \lor (E \supset \neg T)\)

It’s a tautology, which you can check for yourself by doing the truth table.

Suppose I am 99% confident that my sister’s middle name is Angela and that I conclude from this that if my sister’s middle name is not Angela, then Godzilla destroyed Tokyo in 1994. Is this a reasonable inference?

My sister’s middle name is Angela.

Thus, if my sister’s middle name is not Angela, then Godzilla destroyed Tokyo in 1994.

\(A\)

\(\neg A \supset G\)

<table>
<thead>
<tr>
<th>A</th>
<th>G</th>
<th>A</th>
<th>\neg A \supset G</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
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<td>T T T T</td>
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<td>F T T T</td>
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<td>F</td>
<td>T F F F</td>
</tr>
</tbody>
</table>

It’s valid.

Similarly, we know that Yuri Gagarin was a cosmonaut. Should we conclude from this that if Yuri Gagarin was born in 1382 AD, he was a cosmonaut?

\(C\)

\(\overline{Y} \supset C\)

It appears so,

<table>
<thead>
<tr>
<th>Y</th>
<th>C</th>
<th>C</th>
<th>\overline{Y} \supset C</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T T T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F T F</td>
</tr>
</tbody>
</table>
It’s valid.
These results are counterintuitive, so we need some explanation of why they seem unreasonable even though the rules of sentential logic deem these results correct.

13.10. Robustness. How can we explain the discrepancy between arguments that are valid when we use ‘⊃’ to translate conditionals, but that are intuitively dumb inferences to make?

A theory proposed by Frank Jackson says that the truth table for English conditionals is the truth table for ‘⊃’. The difference between them is that the English conditional expresses something else in addition to truth.

There are two things we express when we utter an English conditional: truth and robustness.

When I say “If A, then C,” I am signaling to you that

(1) I believe the sentence A ⊃ C is true.

(2) I will still believe the sentence A ⊃ C is true even if I come to find out that A is true. If this condition holds, we say that A ⊃ C is robust with respect to (learning that) A.

With regard to the argument,

*My sister’s middle name is Angela.*

Thus, if my sister’s middle name is not Angela, then Godzilla destroyed Tokyo in 1994.

although the conclusion follows validly using classical logic, the conclusion is not robust because even if I find out that my sister’s middle name is Beth, I am going to disbelieve the premise of the argument, and then it will not lead me to accepting the conclusion that Godzilla destroyed Tokyo in 1994 if my sister’s middle name is not Angela.

With regard to the argument,

*Yuri Gagarin was a cosmonaut.*

Thus, if Yuri Gagarin was born in 1382 AD, he was a cosmonaut.

Although the conclusion follows validly and I therefore believe it is true, it is not robust because if I find out (to my surprise) that Yuri Gagarin was born in 1382 AD, I am going to use my background knowledge that space travel didn’t exist in 1382, and conclude that the premise is false. Then, the argument will not motivate me to accept the conclusion that if Yuri Gagarin was born in 1382 AD, he was a cosmonaut.

Frank Jackson’s theory isn’t a cheap trick cooked up to fix the paradoxical of the material conditional. It works for disjunction too, as we saw earlier.

Insofar as truth is concerned, an indicative conditional is a disjunction:

\[(A ⊃ B) \equiv (\sim A \lor B)\]

Ordinary language disjunctions are expected to be robust with respect to the denial of either disjunct. If that were translated over to conditionals, a conditional should be robust with respect to the truth of the antecedent and the falsity of the consequent.

If signaling robustness is an important component of language, why don’t we have a linguistic device for signaling robustness with respect to some chosen
proposition $B$? We do. It is called even if '. "A even if $B$" means $A$ is true and you should continue to have confidence in the truth of $A$ if you find out $B$ is true.

Ordinarily we expect people to obey the robustness conditions:

- "$A$ or $B$" is robust with respect to finding out $\neg A$, and is robust with respect to finding out $\neg B$.
- "If $A$ then $B$" is robust with respect to finding out $A$.
- "Even if $A$, $B$" is robust with respect to finding out $A$.
- "Unless $A$, $B$" is robust with respect to finding out $\neg A$.

We often don ’t use the following argument form:

$$\begin{align*}
G \\
\text{If } F, \text{ then } G.
\end{align*}$$

There are two theories that try to explain this:

1. Paul Grice says we often don ’t do it because of conversational conventions. If we are in a position to be more informative, we ought to be more informative. Saying "If $F$, then $G$" is generally less informative than saying "$G$" in cases where $G$ is known to be true.

2. Frank Jackson says we often don ’t do it because the conclusion is often not robust. If you believe $G$, there is no reason in general to believe "If $F$, then $G$" is robust.

13.11. Assertibility. Is the following tautologous, contingent, or contradictory? "If Monica is Italian, then she isn ’t Italian."

<table>
<thead>
<tr>
<th>$M$</th>
<th>$M \supset \neg M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
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</table>

The truth table says it is contingent. It sounds contradictory but it is not according to the truth functional treatment of the if-then connective because if Monica is not Italian, then the conditional doesn ’t apply, and we give the benefit of the doubt to the person saying it.

The puzzle to be explained is why "If Monica is Italian, then she isn ’t Italian," sounds contradictory, when it really isn ’t.

Ernest Adams (1965) proposed a theory that tries to explain the paradoxes of the material conditional. He says that there is a number we can associate with every sentence that represents how willing I am to assert the sentence. This number is called the assertibility. If I am totally confident in saying a certain sentence, we say that the sentence has assertibility 1 for me. If I am totally confident that it is wrong to say it, we say that the sentence has assertibility 0 for me. If I am unsure of whether I should say the sentence, then it has assertibility somewhere between 0 and 1.

For most sentences, the assertibility is equal to how likely I think that the sentence is true.

The idea is that I am willing to state a sentence to the degree I think it is true. In order to model assertibility, we abstract away from all the various contexts that might affect how willing I am to assert a sentence. For example, I may not want to say something because it is a secret, or because it is embarrassing, or because it would be boring, etc. But we don ’t count any of these kinds of factors.
The trick that helps us resolve the paradoxes is that the assertibility of a conditional is not equal to how likely I think that it is true. Instead, it is equal to how likely the consequent is true, given that the antecedent is true.

To figure out the assertibility of “If A, then B,” I assume A, and then figure out how likely it is that B.

Example: “If it is raining, then it is sunny.” To reckon its assertibility, I take for granted that it is raining, and then think about how likely it is that it is sunny. Under most circumstances this will be a pretty low number.

To figure the truth of “If A, then B,” I do the truth table. When it’s not raining this will automatically be true even if there is a good reason to think raininess makes sunniness much less likely (or even impossible).

What Ernest Adams showed was that the inferences we find intuitively compelling are exactly those that preserve high assertibility, not inferences that are logically valid.

To find out whether an argument preserves assertibility, we look for a situation where all the premises have high assertibility, but where the conclusion has very low assertibility. If there is such a situation, the argument does not preserve assertibility. If not, the argument does preserve assertibility.

This is exactly parallel to our definition of validity.

What Ernest Adams showed was that the inferences we find intuitively compelling are exactly those that preserve high assertibility, not inferences that are logically valid.

Examples where the assertibility-based treatment matches the truth-based treatment.

Modus Ponens:

*If unicorns exist, then goblins exist too.*

*Unicorns do exist.*

*Thus, goblins exist.*

Logically valid and preserves assertibility.

Modus Tollens:

*If unicorns exist, then goblins exist too.*

*Goblins don’t exist.*

*Thus, unicorns don’t exist.*

Logically valid and preserves assertibility.

Affirming the Consequent:

*If unicorns exist, then goblins exist too.*

*Goblins exist.*

*Thus, unicorns exist.*

Logically invalid and does not preserve assertibility.

Denying the Antecedent:

*If unicorns exist, then goblins exist too.*

*Unicorns don’t exist.*

*Thus, goblins don’t exist.*

Logically invalid and does not preserve assertibility.

Examples where the assertibility-based treatment is superior to the truth-based treatment.
The statement “If A, then ¬A” is not a contradiction according to the truth-based semantics. The assertibility-based semantics explains why it sounds like a contradiction by noting that all arguments of this form have zero assertibility no matter how things stand in the world.

Adams defined the probability conditional $A \Rightarrow C$, adding it to the other rules of sentential logic: $(\alpha \Rightarrow \beta)$ is a wff whenever $\alpha$ is a wff and $\beta$ is a wff. But a formula with somewhere other than the main connective is not a wff. So $\neg(A \Rightarrow C)$, $(A \Rightarrow (B \Rightarrow C))$, and $(A \vee (B \Rightarrow C))$ are not wffs.

Adams then defined probabilistic validity: The probability of statements without any $\Rightarrow$ is the probability of their truth. The probability of statements of the form $(\alpha \Rightarrow \beta)$ is the conditional probability of $\beta$ given $\alpha$. This quantity is expressed as $p(\beta/\alpha)$. A probabilistic counterexample is a situation where the probability of every premise is very high, and the probability of the conclusion is very low. An argument is probabilistically valid iff it has no probabilistic counterexamples.

Let us now examine several inferences that are valid with the material conditional to see if they are also valid with the probabilistic conditional.

Modus Ponens: Modus ponens for $\supset$ is definitely valid, and it was long believed to be valid for the English indicative. But what about the following?

$$
\begin{align*}
A \Rightarrow B \\
A \\
B
\end{align*}
$$

Disjunctive Syllogism: Disjunctive syllogism for $\supset$ is definitely valid, and it was long believed to be valid for the English indicative. But what about the following?
2. PROPOSITIONAL LOGIC

Figure 2. Counterexample to disjunctive syllogism

\begin{align*}
A \land B \\
\neg A \Rightarrow B
\end{align*}

The sun will continue to exist for the next year or the weather will be warm the year after.

If the sun does not continue to exist for the next year, the weather will be warm the year after.

Contraposition: Contraposition for ‘⇒’ is definitely valid, and it was long believed to be valid for the English indicative. But what about the following?

\begin{align*}
A \Rightarrow B \\
\neg B \Rightarrow \neg A
\end{align*}

If the sun rises tomorrow, then it will warm up during the day.

If it does not warm up during the day (tomorrow), the sun will not rise tomorrow.

If I have any children at all, I don’t have any more than two.

If I have more than two children, I don’t have any children at all.

If you want beer, I have some in the fridge.

If I don’t have any beer in the fridge, you don’t want beer.
Modus Tollens: Modus tollens for ‘⊃’ is definitely valid, and it is still believed to be valid for the English indicative. But what about the following?

\[
A \Rightarrow B \\
\neg B \\
\neg A
\]

*If I have any children at all, I don’t have more than two.*  
*If I do have more than two children.*  
*I don’t have any children at all.*

Antecedent Strengthening: Antecedent strengthening for ‘⊃’ is definitely valid, and it was long believed to be valid for the English indicative. But what about the following?

\[
B \Rightarrow C \\
(\neg A \& B) \Rightarrow C
\]

*If the egg is dropped, it will break.*  
*If the egg is packed in Styrofoam and is dropped, it will break.*
Hypothetical Syllogism: Hypothetical syllogism for ‘⊃’ is definitely valid, and it was long believed to be valid for the English indicative. But what about the following?

\[ A \Rightarrow B \]
\[ B \Rightarrow C \]
\[ A \Rightarrow C \]

Modus Ponens with a Conditional Consequent: All cases of modus ponens for ‘⊃’ are definitely valid. But what about the following?

\[ A \Rightarrow (B \Rightarrow C) \]
\[ A \]
\[ B \Rightarrow C \]

If a Republican wins, then if Reagan doesn’t win, Anderson wins.
A Republican will win.
If Reagan doesn’t win, Anderson will win.

This example illustrates that indicative conditionals serve as devices to restrict consideration to the set of possibilities countenanced in the antecedent.

Recall that in the Adams’ logic, ‘⇒’ cannot appear anywhere except as the main connective.

But, it is possible that \( A \Rightarrow (B \Rightarrow C) \) can be thought of as a misleading way of stating \( (A \& B) \Rightarrow C \). Exportation and importation would be true by stipulating that the meaning of the ‘⇒’ embedded in the consequent makes it true.

This would block the problematic inferences McGee found.

Conditionals can be assigned at least two kinds of semantic values: truth and assertibility. We can do logic only considering the truth aspect. But our intuitions about what seems reasonable take into account assertibility (subjective probability). Thus, we will get a lot of logical results with material conditionals that sound funny, but which make sense if we remember that indicative conditionals have assertibility conditions and obey a probabilistic logic.

It would be nice if these two pictures could be integrated with one another by assuming indicative conditionals have truth conditions and interpreting assertibility as the probability that the indicative conditional is true.
Interestingly, there are some proofs called the triviality results that show that linking the truth and assertibility of indicative conditionals in such a way can only be done in a small number of special cases.

The semantic value of $A \Rightarrow B$ (its conditional probability) is not the probability that $A \Rightarrow B$ is true. The semantic value appears to be sui generis.

The delinking of truth and assertibility suggests that humans use a patchwork of reasoning principles.

For purposes of reasoning with conditionals, we tend to find reasonable those inference forms that preserve assertibility.

When embedding conditionals inside more complex expressions or when referencing them, we tend to think of them in terms of truth.

Wendy says, “There are 27 parts in the toy. If a part is blue, it’s spherical. If a part is red, it’s cubic.” Evan says, “Wendy made three claims, only two of which are true.”

Either [there are 27 parts in the toy and if a part is blue, then it is spherical,] or [there are 27 parts in the toy and if a part is red, then it is cubic,] or [if a part is blue, then it is spherical and if a part is red, then it is cubic.]

13.12. Application to Ethics. A famous principle of ethical theory is that it is always illegitimate to derive an ‘ought’ from an ‘is’. In other words, one can never legitimately infer a normative claim from merely descriptive claim. No amount of information describing how nature behaves materially is sufficient to inform us about whether some goal is noble or whether some action is just or praiseworthy or morally permissible, etc. unless it is augmented with some normative claim such as a moral principle.

Arthur Prior in “The Autonomy of Ethics” claimed that one can derive normative propositions from purely descriptive propositions.

Here are two examples:

Tea-drinking is common in England.

Thus, either tea-drinking is common in England or all New Zealanders ought to be shot.

Notice that in this example, the inference is valid and probabilistically valid, but does not preserve robustness. So, perhaps, robustness preservation needs to be understood as part of the implied logic that underwrites the prohibition against deriving and ‘ought’ from an ‘is.’ Also notice that in order for the disjunction to be actionable in a particular case, i.e. as a stand-alone ethical claim, it (in effect) needs to have the descriptive disjunct denied (so that disjunctive syllogism can be employed). But to the extent one embraces the negation of the descriptive disjunct, one rejects the descriptive disjunct used to arrive at the disjunction. So, it appears the (disjunctive) ethical principle cannot have any bearing on one’s behavior.

Undertakers are Church officers.

Thus, undertakers ought to do whatever all Church officers ought to do.

Notice that

\[
\begin{array}{c}
D_1 \\
\land \\
D_2 \\
\hline
N_1 \\
N_2
\end{array}
\]

implies
\begin{align*}
\frac{D_1 \& D_2}{N_1 \supset N_2}
\end{align*}
and
\begin{align*}
\frac{D_1 \& D_2}{N_1 \Rightarrow N_2}
\end{align*}

So when the $D_i$’s are purely descriptive claims, and the $N_i$’s are normative claims, we can convert an argument that is essentially normative in character, into one that formally has only descriptive premises. The issue about the successfulness of Prior’s counterexample comes down to whether arguments of the second kind here (with a conditional premise) are implicitly arguments of the first kind (with an unconditional premise). In general, they are not, nor does the validity or probabilistic validity of the second argument imply (or probabilistically imply) the validity or probabilistic validity of the first argument. However, one might argue that the important upshot of the ‘is’ does not imply ‘ought’ injunction is that we are unable to derive a unconditional moral claim, something like, “I ought to share” or “She needs to apologize.” In order to get to a conclusion that is normative but does not get its normativity solely by being a conditional with a normative antecedent, normative premises are needed.

Conclusion: In the first example, the inference is logically valid, but is not probabilistically valid. In the second example, the inference results in a conditional conclusion, which in effect rewrites an argument with a normative premise into an argument with merely descriptive premises and the normative claim shifted into the antecedent of the conditional conclusion. Thus, Prior’s counterexamples require us to be sophisticated about how we should understand the notion of inference relevant to the prohibition against inferring an ‘ought’ from an ‘is’, but it does not threaten the core content of the principle—that one cannot legitimately derive an actionable ‘ought’ from merely descriptive information.
CHAPTER 3

Truth Trees

Truth trees (also called semantic tableaux) are a method of determining consistency for sets of statements. They are usually more efficient than truth tables, depending on the sentences involved. Another benefit of using truth trees over truth tables is that we can continue to use them when we expand our logic to include quantifiers.

In our system there are 7 rules, illustrated in Fig. 1.

(1) **Double Negation** From any sentence of the form \( \neg \neg \alpha \), derive \( \alpha \).
(2) **Conjunction** From any sentence of the form \( \alpha \& \beta \), derive \( \alpha \) and derive \( \beta \).
(3) **Negation of Conjunction** From any sentence of the form \( \neg (\alpha \& \beta) \), create two branches, one for \( \neg \alpha \) and another for \( \neg \beta \).
(4) **Disjunction** From any sentence of the form \( \alpha \lor \beta \), create two branches, one for \( \alpha \) and another for \( \beta \).
(5) **Negation of Disjunction** From any sentence of the form \( \neg (\alpha \lor \beta) \), derive \( \neg \alpha \) and derive \( \neg \beta \).
(6) **Conditional** From any sentence of the form \( \alpha \supset \beta \), create two branches, one for \( \neg \alpha \) and another for \( \beta \).
(7) **Negation of Conditional** From any sentence of the form \( \neg (\alpha \supset \beta) \), derive \( \alpha \) and derive \( \neg \beta \).

To create the truth tree, we follow these rules:

(1) Start by vertically listing all the statements whose consistency you want to check.
(2) Pick any complex statement you like that is in an open branch of the tree, and apply the rule appropriate to that statement. Strategy Tip: Do the non-branching rules first.
(3) Check for explicit inconsistencies along branches of the tree. An explicit inconsistency is when a formula and its negation appear on the same branch. If you find an explicit inconsistency, close off the branch by writing an \( \times \) at the bottom of the branch where the inconsistency appears.
(4) Go to step two, unless either all branches are closed, in which case the set is inconsistent, or there is a branch where you have run out of complex statements, in which case the set is consistent.
(5) Number every line that has a statement.
(6) Mark each derived line with the number of the line from which you derived it and the rule you used.
(7) Mark at the bottom whether the set is consistent or inconsistent.

If we are checking for the validity of an argument, remember that we initially list the negation of the conclusion along with the premises. A fully closed tree
Double Negation
\[\sim\sim\alpha\]
\[\sim\alpha\]

Conjunction
\[(\alpha \& \beta)\]
\[\alpha\]
\[\beta\]

Negation of Conjunction
\[\sim(\alpha \& \beta)\]
\[\sim\alpha\]
\[\sim\beta\]

Negation of Conditional
\[\sim(\alpha \supset \beta)\]
\[\sim\alpha\]
\[\beta\]

Negation of Disjunction
\[\sim(\alpha \vee \beta)\]
\[\sim\alpha\]
\[\sim\beta\]

Conditional
\[(\alpha \supset \beta)\]
\[\sim\alpha\]
\[\beta\]

Disjunction
\[(\alpha \vee \beta)\]
\[\alpha\]
\[\beta\]

**Figure 1.** Sentential Truth Tree Rules

means the argument is valid, and a tree with an open branch means the argument is invalid. Counterexamples can be found by looking at the atomic sentences on any one open branch.

**1. Truth Tree Examples**

**Exercise 12.** Use a truth tree to determine whether the argument is valid.

\[W \lor \sim(T \& U)\]
\[\sim T\]
\[\sim(W \& \sim Y)\]

One counterexample to the argument in exercise 12 (corresponding to the leftmost branch of the tree) is a world where \(W\) and \(T\) are true and \(Y\) is false.

**Exercise 13.** Use a truth tree to determine whether the argument is valid.

\[\sim J \lor \sim K\]
\[L \& \sim M\]
\[(K \& L) \supset (J \lor O)\]
\[O\]

The only counterexample to the argument in exercise 13 is where \(L\) is true and \(O\), \(M\), and \(K\) are false.

**Exercise 14.** Use a truth tree to determine whether the argument is valid.

\[E \supset \sim(D \& C)\]
\[(E \lor C) \supset D\]
\[B \& \sim C\]
\[D \supset (B \& C)\]
Exercise 15. Use a truth tree to determine whether the argument is valid.

\[
\begin{align*}
(F \& G) \lor (H \& I) \\
(G \& H) \lor (F \& I) \\
(G \& I) \lor (F \& H)
\end{align*}
\]
3. TRUTH TREES

Figure 4. Answer to Exercise 14

Figure 5. Answer to Exercise 15
1. The Quantifier Language

The language of quantifier logic is built on that of Sentential (Propositional) Logic.

(1) First, we add lowercase letters to our language.
(2) Constants: lowercase letters from ‘a’ to ‘t’ will refer to individuals.
(3) Variables: lowercase letters from ‘u’ to ‘z’ will be variables that will only appear within the scope of a quantifier symbol.
(4) Second, we add uppercase letters that are immediately followed by lowercase letters, e.g. $P_x, Rab, Bycz$, and these denote properties and relations.
(5) Third, we add quantifier symbols $\forall$ and $\exists$, which are always followed by a single variable (or in the shorthand that will be allowed, multiple distinct variables).

Other logic systems mark the universal quantifier by putting parentheses around the variable: $(x)(Mx \supset Ax)$ and the existential quantifier as $(\exists y)(Cy \& By)$.

With the new elements of our language, we need new rules for wffs.

- Any capital letter is a wff.
- Any capital letter immediately followed by $n$ lowercase letters (either individuals or variables) is a wff.
- If $\alpha$ is a wff, then $\neg \alpha$ is a wff.
- If $\alpha$ is a wff, and $\beta$ is a wff, then $(\alpha \& \beta)$ is a wff.
- If $\alpha$ is a wff, and $\beta$ is a wff, then $(\alpha \lor \beta)$ is a wff.
- If $\alpha$ is a wff, and $\beta$ is a wff, then $(\alpha \rightarrow \beta)$ is a wff.
- If $\alpha$ is a wff and $x$ is a variable, then $\forall x \alpha$ is a wff.
- If $\alpha$ is a wff and $x$ is a variable, then $\exists x \alpha$ is a wff.
- Nothing else is a wff.

As shorthand, we can abbreviate $\forall x \forall y \forall z$ as $\forall xyz$, and $\exists w \exists x$ as $\exists wx$. It doesn’t matter if you switch the order of the variables.

We need a rule to define the connection between wffs and sentences. We say that an instance of a variable is **bound** if it is inside the scope of a quantifier with the same kind of variable (same letter). Otherwise we say that the instance of the variable is **free**. A wff is a sentence if and only if all its variables are bound.

**Exercise 16. Which of the following are sentences?**

1. $\forall x(My \supset Ax)$
2. $\neg \exists z(Mz \& Ah) \& Mz$
3. $\exists x \forall y(Ky \supset (\neg Ax \& \neg Q))$
4. $\exists x \exists z(Rxz \& Ex)$
5. $\forall x \forall y \forall z(Rxy \supset \neg Hj)$
(6) \( \exists x \forall z (Wx \supset Sd) \)
(7) \( \forall z (Rz \supset \neg \neg Fz) \supset \neg \forall x (Rz \supset Sx) \)
(8) \( \exists x \forall z (Cxz \supset Cxz) \)
(9) \( \forall y x \neg (My \supset Adxd) \)
(10) \( \exists z (Fz \& Gz) \supset \exists z (Rz \supset Sz) \)
(11) \( \exists x (Tx \& \forall x (Cxx \supset K)) \)

Answers: 3, 4, 5, 6, 9, 10, 11. In 1, \( y \) is free. In 2, the \( z \) at the end is not within the scope of a quantifier. In 7, the rightmost \( z \) is free. 8 is not even a wff due to the negation sign in the wrong place. 6 and 11 have quantifiers that don’t do anything because the quantified variable does not appear inside the scope of the quantifier, but they are legitimate sentences nonetheless.

2. Predicate Logic Truth Tree Rules

Negated Quantifier Rules:
(-\( \exists \)): From \( \neg \exists x (\alpha) \), derive \( \forall x (\neg \alpha) \).
(-\( \forall \)): From \( \neg \forall x (\alpha) \), derive \( \exists x (\neg \alpha) \).

Existential Quantifier Rule (\( \exists \)):
From \( \exists x (\alpha) \), derive \( \alpha \) with every instance of \( x \) in \( \alpha \) replaced by a new constant.

Universal Quantifier Rule (\( \forall \)):
From \( \forall x (\alpha) \), derive \( \alpha \) with every instance of \( x \) in \( \alpha \) replaced by a constant. Don’t cross out the line with \( \forall x (\alpha) \).

Strategies:
- Your first priority is to push any negations inside the quantifiers. Use the \( \neg \exists \) and \( \neg \forall \) rules to move negations from the left side of quantifiers to the right side.
- Your second priority is to do the \( \exists \) rule before the \( \forall \) rule whenever possible.
- When you use \( \forall \) on a statement, pick individuals that already appear in its branch of the truth tree, and target those choices so that the instantiated variable matches an existing constant. For example, if you already have \( \text{Rakh} \) in your branch, and you are instantiating something of the form \( \forall x ((Ryxz \& \ldots) \supset) \), you want to instantiate with \( k \).
- You should always aim to close all the branches.

The big difference created by adding quantifiers to the previous rules for truth trees is that sometimes you will be able to keep applying the rules forever. Some trees will not close even though you have more rules available to apply. This means at some point, you need to make an educated guess as to whether the tree will go on no matter which rules you select. If the tree is destined to go on forever, it is considered open.

This difference between propositional and quantifier logic is important. We say propositional logic is decidable and quantifier logic is undecidable. For a logic to be decidable means that for any sentence \( S \) expressed in that logic, it is always possible to decide whether \( S \) is a contradiction. A decidable logic can also be defined as a logic where it is always possible to decide whether \( S \) is a theorem, i.e. a necessary truth.

3. Predicate Translations

The way to translate the sentence, “Bill is acting odd.” is to let
3. PREDICATE TRANSLATIONS

\[
b = \text{Bill}\\
Ox = x \text{ is acting odd.}
\]
and write the sentence as \( Ob \).

The way to translate the sentence, “Fiona will be at home sometime during the next week,” is to let

\[
f = \text{Fiona}\\
Hx = x \text{ will be at home sometime during the next week.}
\]
and write the sentence as \( Hf \).

The way you do the translation is to check whether the subject of the sentence refers to an individual. If it does, translate the subject as a lowercase letter. If not, you will have to fall back on the other rules we have for translation in propositional logic. Assuming the subject of the sentence is an individual, translate what is said about that individual as a property. This will work for just about any simple sentence, even if you don’t normally think of the sentence as talking about the properties of individuals. It also works for a lot of complicated sentences.

“The most repulsive man on Earth won the shock jock’s prize.” translates as

\[
r = \text{the most repulsive man on Earth}\\
Wx = x \text{ won the shock jock’s prize.}\\
Wr
\]

“The only person on the bench was sleeping.” translates as

\[
p = \text{the only person on the bench}\\
Sx = x \text{ was sleeping.}\\
Sp
\]

“Rosco should have done it.” translates as

\[
r = \text{Rosco}\\
Sx = x \text{ should have done it.}\\
Sr
\]

“Randy groomed his pet.” translates as

\[
r = \text{Randy}\\
P x = x \text{ groomed } x\text{’s pet.}\\
Pr
\]

Of course, it’s not always that easy. If the sentence has the kind of logical structure that we saw in propositional logic, we have to capture that as well. The technique used is exactly the technique that you have already learned. First, reformulate the sentence by separating independent clauses with ‘and’ and ‘or’ and pull negations out of the sentence where possible with ‘it is not the case that’. Then, convert sentences with individuals as subjects into properties and individuals.

“The last person in line won’t get into the show.” translates as, “It is not the case that the last person in line will get into the show.” which translates as

\[
l = \text{the last person in line.}\\
Sx = x \text{ will get into the show.}\\
\neg Sl
\]

“Dave and Martha are top prize winners in this year’s competition.” translates as “Dave is a top prize winner in this year’s competition, and Martha is a top prize winner in this year’s competition.” which translates as
d = Dave  
m = Martha  
\( W x = x \) is a top prize winner in this year’s competition.  
\( W d \& W m \)

“The horseman approached us quickly but silently,” translates as “The horseman approached us quickly, and the horseman approached us silently.” which translates as

\[ h = \text{the horseman} \]
\[ Q x = x \text{ approached us quickly.} \]
\[ S x = x \text{ approached us silently.} \]
\[ Q h \& S h \]

“The governor was lambasted and ridiculed for what he said.” translates as “The governor was lambasted for what he said, and the governor was ridiculed for what he said.” which translates as

\[ g = \text{the governor} \]
\[ L x = x \text{ was lambasted for saying what he or she said.} \]
\[ R x = x \text{ was ridiculed for saying what he or she said.} \]
\[ L g \& R g \]

“Janie and Gillian were pleased with the decision, but Ed wasn’t.” translates as “Janie was pleased with the decision, and Gillian was pleased with the decision, and it is not the case that Ed was pleased with the decision.” which translates as

\[ j = \text{Janie} \]
\[ g = \text{Gillian} \]
\[ e = \text{Ed} \]
\[ P x = x \text{ was pleased with the decision.} \]
\[ (P j \& P g) \& \neg P e \]

“If Elena ate the rice I left in the fridge, she is going to be sick.” translates as

\[ e = \text{Elena} \]
\[ A x = x \text{ ate the rice I left in the fridge.} \]
\[ S x = x \text{ is going to be sick.} \]
\[ A e \supset S e \]

“It is not the case that both Ray and his brother are unemployed,” translates as “It is not the case that (it is not the case that Ray is employed and it is not the case that Ray’s brother is employed).”

\[ r = \text{Ray} / / b = \text{Ray’s brother} \]
\[ E x = x \text{ is employed.} \]
\[ \neg (\neg E r \& \neg E b) \]

“The engine and caboose are painted in solid blue and solid red respectively,” translates as “The engine is painted in solid blue, and the caboose is painted in solid red.”

\[ e = \text{the engine} \]
\[ c = \text{the caboose} \]
\[ B x = x \text{ is painted in solid blue.} \]
\[ R x = x \text{ is painted in solid red.} \]
\[ B e \& R c \]
4. QUANTIFIERS

Remember that you can translate a sentence into the individual w/ property form only if there is an individual. If the subject of the sentence is not an individual, you have to translate the sentence in the familiar propositional way.

“Goats eat everything, but my pet is a very picky eater.” translates as

\[ G = \text{Goats eat everything.} \]
\[ p = \text{my pet} \]
\[ Px = x \text{ is a picky eater} \]

\[ G \& Pp \]

“I’m not a hero if heroes always do what’s right.” translates as

\[ R = \text{Heroes always do what’s right.} \]
\[ m = \text{me} \]
\[ Hx = x \text{ is a hero.} \]

\[ R \supset \neg Hm \]

4. Quantifiers

We want our improved logic to capture the logic of some expressions that say something about the number (quantity) of things that have properties.

(1) “All dogs have fur.”
(2) “Some flowers are fragrant.”
(3) “At least one singer is blond.”
(4) “Every instrument in the band is made of brass.”
(5) “Whatever person is late will be left behind.”
(6) “Anyone who objects will be silenced.”
(7) “Nothing is going to happen without approval.”
(8) “No one likes to have their eyes poked.”
(9) “Not everyone is going to go for that idea.”

The fundamental concepts behind these sentences are the universal quantifier that expresses ‘every’ or ‘all’ and the existential quantifier that expresses ‘at least one’.

Universal Quantifier \( \forall \) means ‘every’.
Existential Quantifier \( \exists \) means ‘at least one’.

These two symbols will allow us to translate a number of different sentences and analyze a surprisingly large set of more sophisticated arguments.

4.1. Quantifiers We Don’t Translate. These expressions make implications about number or quantity but are too complicated for our simple quantifier logic to handle.

(1) “Most sorority members use highlights.”
(2) “A few limbs were damaged in the accident.”
(3) “Few would offer that opinion.”
(4) “Phoenicians were by and large sailors.”
(5) “The overwhelming number of congressmen are rich.”
(6) “The number of days in the week exceeds the number of quark flavors by one.”
(7) “Birds fly.”
(8) “Many people have tried and failed.”
5. Universal Sentences

Universal sentences are translated with ∀. Some extremely simple sentences, sentences that make a claim about absolutely everything, are translated with ∀ followed by a property.

“Everything is ephemeral.”
Ex = x is ephemeral.
∀xEx

“All is lost.”
Lx = x is lost.
∀xLx

“Everything is black and dreary.”
Bx = x is black.
Dx = x is dreary.
∀x(Bx&Dx)

“Everything is either real or isn’t real.”
Rx = x is real.
∀x(Rx ∨ ¬Rx)

For almost all universal sentences, it’s slightly more complicated. Step 1 is to translate into a conditional with the following idiom: “Every thing is such that if it is... then it is...”

• “All dogs have fur,” translates as “Every thing is such that if it is a dog, then it has fur.”
• “Every story has embellishments,” translates as “Every thing is such that if it is a story, then it has embellishments.
• “Every tree has leaves or needles,” translates as “Every thing is such that if it is a tree, then (it has leaves or it has needles).”
• “Everyone is thrilled,” translates as “Every thing is such that if it is a person, then it is thrilled.”
• “Everyone who’s late will be turned away,” translates as “Every thing is such that if it is a person and it is late, then it will be turned away.”
• “All fallen fruit rots,” translates as “Every thing is such that if it is a fruit and it is fallen, then it rots.”

Step 2 is to translate into the form ∀x(conditions on the subject ⊃ property).”

Every moose has antlers.
Mx = x is a moose.
Ax = x has antlers.
∀x(Mx ⊃ Ax)

“All dogs have fur,” translates as “Every thing is such that if it is a dog, then it has fur.”

Dx = x is a dog.
Fx = x has fur.
∀x(Dx ⊃ Fx)

“Every story has two sides” translates as “Every thing is such that if it is a story, then it has two sides.”

Sx = x is a story.
$Tx = x$ has two sides.
$\forall x (Sx \supset Tx)$

“Every tree has leaves or needles,” translates as “Every thing is such that if it is a tree, then (it has leaves or it has needles).”
$Tx = x$ is a tree.
$Lx = x$ has leaves.
$Nx = x$ has needles.
$\forall x (Tx \supset (Lx \lor Nx))$

“Everyone is thrilled,” translates as “Every thing is such that if it is a person, then it is thrilled.”
$Px = x$ is a person.
$Tx = x$ is thrilled.
$\forall x (Px \supset Tx)$

“Everyone who’s late will be turned away,” translates as “Every thing is such that if it is a person and it is late, then it will be turned away.”
$Px = x$ is a person.
$Lx = x$ is late.
$Tx = x$ will be turned away.
$\forall x((Px \& Lx) \supset Tx)$

“All fruit on the ground rots,” translates as “Every thing is such that if it is a fruit and it is fallen, then it rots.”
$Fx = x$ is a fruit.
$Gx = x$ is on the ground.
$Rx = x$ rots.
$\forall x((Fx \& Gx) \supset Rx)$

“All mud huts without permits will be condemned and demolished,” translates as “Every thing is such that if it is a hut, and it is made of mud, and it is not the case that it has a permit, then it will be condemned and it will be demolished.”
$Hx = x$ is a hut.
$Mx = x$ is made of mud.
$Px = x$ has a permit.
$Cx = x$ will be condemned.
$Dx = x$ will be demolished.
$\forall x(((Hx \& Mx) \& \neg Px) \supset (Cx \& Dx))$

“Every indicted suspect in this case will have his or her fingerprints on file,” translates as “Every thing is such that if it is indicted and it is a suspect in this case, then it will have its fingerprints on file.”
$Ix = x$ is indicted.
$Sx = x$ is a suspect in this case.
$Fx = x$ will have its fingerprints on file.
$\forall x((Ix \& Sx) \supset Fx)$

“If everyone agrees, Jack will get started,” translates as “If (every thing is such that if it is a person, then it agrees), then Jack will get started.”
$j = \text{Jack}$
$Px = x$ is a person.
$Ax = x$ agrees.
$Sx = x$ will get started.
$\forall y (Py \supset Ay) \supset Sj$

6. Existential Sentences

Existential sentences are translated with $\exists$. Some simple sentences, sentences that make a claim asserting the existence of something, are translated with $\exists$ followed by a property.

“There exists a demon.”
$Dx = x$ is a demon.
$\exists x Dx$

“Something is lost.”
$Lx = x$ is lost.
$\exists x Lx$

“There is at least one universe.”
$Ux = x$ is a universe.
$\exists x Ux$

More common sentences make more restricted claims of existence. For these, one forms a large conjunction of all the properties of the subject whether they come from adjectives modifying the subject or are predicates following the verb.

1. Translate into a conjunction with the following idiom: “There exists a thing such that (it is..., and it is..., and it is..., etc.).”

2. Translate the conjunction into symbols: $\exists x ((Ax \& Bx) \& (Cx \& (Dx \& Ex)))$

“Some castle wall was breached,” translates as “There exists a thing such that (it is a castle wall and it is breached).”
$Cx = x$ is a castle wall.
$Bx = x$ was breached.
$\exists y (Cy \& By)$

“There’s a ferret in my bed,” translates as “There exists a thing such that (it is a ferret and it is in my bed).”
$Fx = x$ is a ferret.
$Bx = x$ is in my bed.
$\exists x (Fx \& Bx)$

“Some child stole the candy on the back shelf of the store,” translates as “There exists a thing such that (it is a child and it stole the candy on the back shelf of the store).”
$Cx = x$ is a child.
$Sx = x$ stole the candy on the back shelf of the store.
$\exists x (Cx \& Sx)$

“A red car was sold yesterday,” translates as “There exists a thing such that (it is a car, and it is red, and it was sold yesterday).”
$Cx = x$ is a car.
$Rx = x$ is red.
$Sx = x$ was sold yesterday.
$\exists z ((Cz \& Rz) \& Sz)$
“Someone underage will be at the strip club and will likely be getting blasted,” translates as “There exists a thing such that (it is a person, and it is underage, and it will be at the strip club, and it will likely be getting blasted).”

\[ P_x = x \text{ is a person.} \]
\[ U_x = x \text{ is underage.} \]
\[ S_x = x \text{ will be at the strip club.} \]
\[ B_x = x \text{ will likely be getting blasted.} \]
\[ \exists y ((P_y \& U_y) \& (S_y \& B_y)) \]

“A rather long python was seen at the scene of the crime,” translates as “There exists a thing such that (it is a python, and it is rather long, and it was seen at the scene of the crime).”

\[ P_x = x \text{ is a python.} \]
\[ L_x = x \text{ is rather long.} \]
\[ S_x = x \text{ was seen at the scene of the crime.} \]
\[ \exists x ((P_x \& L_x) \& S_x) \]

“Someone is to blame for this mess, and it’s not me,” translates as “There exists a thing such that (it is a person, and it is to blame for this mess), and it is not the case that I am to blame for this mess.

\[ P_x = x \text{ is a person.} \]
\[ B_x = x \text{ is to blame for this mess.} \]
\[ m = \text{me} \]
\[ \exists x ((P_x \& B_x) \& \neg B_m) \]

“Unicorns exist,” translates as “There exists a thing such that it is a unicorn.”

\[ U_x = x \text{ is a unicorn.} \]
\[ \exists x U_x \]

We don’t worry about the fact that ‘unicorns’ is plural because the sentence can plausibly be interpreted as true if a single unicorn is discovered.

6.1. Any, Anyone, Anybody, Anything. Most of the time, ‘any’ is translated as a universal but sometimes as an existential. The way to tell is just to try to substitute ‘everyone’ and try to substitute ‘something’ and go with what sounds like a better translation. Tip: The existential translation usually comes in the antecedent of some conditional.

Suppose we are translating, “Anyone can ride a bike.” We ask ourselves which sounds more accurate: “Everyone can ride a bike,” or “Someone can ride a bike”? “Everyone can ride a bike,” sounds better so we go ahead and translate that to “Every thing is such that (if it is a person then it can ride a bike).” and then translate that directly into symbols as \[ \forall x (P_x \supset B_x). \]

With “If anyone comes, I will know,” we ask ourselves which sounds more accurate: “If everyone comes, I will know,” or “If someone comes, I will know”? Because “If someone comes, I will know,” sounds better, we take it and translate it into, “If (there exists a thing such that it is a person and that thing comes), then I will know,” and then into symbols as \[ \exists x (P_x \& C_x) \supset K. \]

With “Anything that exposes us to risk will be handled with aggression,” we we ask ourselves which sounds more accurate: “Everything that exposes us to risk will
be handled with aggression,” or “Something that exposes us to risk will be handled with aggression.” We use the ‘everything’ translation and spell it out as “Everything is such that (if it exposes us to risk, then that thing will be handled with aggression),” and then as $\forall x (Rx \supset Ax)$.

With “Not just anyone can fly an airplane,” we ask ourselves which sounds more accurate: “Not everyone can fly an airplane,” or “Not someone can fly an airplane.” (We eliminate the ‘just’ because it does not seem to contribute to the meaning of the sentence, and it makes the ‘everyone’ version and ‘someone’ version awkward.) The ‘someone’ version says no one can fly an airplane, so we use the first version and expand it into, “It is not the case that (everyone can fly an airplane),” and then into $\neg \forall x (Px \supset Fx)$.

6.2. No, None, No one, Nobody, Nothing. These expressions translate as follows:

- ‘no’ = ‘not some’
- ‘none’ = ‘not one’
- ‘no one’ = ‘not someone’
- ‘nobody’ = ‘not somebody’
- ‘nothing’ = ‘not something’

This means they are negations of existential claims.

“No dogs are allowed,” translates as “Not some dogs are allowed,” which translates as “It is not the case that (there exists a thing such that (it is a dog and it is allowed)).”

$$Dx = x \text{ is a dog.}$$
$$Ax = x \text{ is allowed.}$$
$$\neg \exists x (Dx \& Ax)$$

“No white buffaloes can be seen today,” translates as “Not some white buffaloes can be seen today,” which translates as “It is not the case that (there exists a thing such that (it is a buffalo and it is white and it can be seen today)).”

$$Bx = x \text{ is a buffalo.}$$
$$Wx = x \text{ is white.}$$
$$Sx = x \text{ can be seen today.}$$
$$\neg \exists x ((Bx \& Wx) \& Sx)$$

“No one here is yelling and screaming,” translates as “Not someone here is yelling and is screaming,” which translates as “It is not the case that (there exists a thing such that (it is a person and it is here and it is yelling and it is screaming)).”

$$Px = x \text{ is a person.}$$
$$Hx = x \text{ is here.}$$
$$Yx = x \text{ is yelling.}$$
$$Sx = x \text{ is screaming.}$$
$$\neg \exists x ((Px \& Hx) \& (Yx \& Sx))$$

“There is nothing that is noble that is worth fighting for,” translates as “There is not something noble that is worth fighting for,” which translates as “It is not the case that (there exists a thing such that (it is noble and it is worth fighting for)).”

$$Nx = x \text{ is noble.}$$
6. EXISTENTIAL SENTENCES

\( Wx = x \) is worth fighting for.
\( \neg \exists x (Nx \& Wx) \)

“Nobody who has put up with your outbursts or has dealt with you for any length of time is going to want to celebrate your engagement,” translates as “Not somebody who (has put up with your outbursts or has dealt with you for any length of time) is going to want to celebrate your engagement,” which translates as “It is not the case that (there exists a thing such that (it is a person and (it has put up with your outbursts or it has dealt with you for any length of time) and it is going to want to celebrate your engagement)).”

\( Px = x \) is a person.
\( Ox = x \) has put up with your outbursts.
\( Dx = x \) has dealt with you for any length of time.
\( Cx = x \) is going to want to celebrate your engagement.
\( \neg \exists x ((Px \& (Ox \lor Dx)) \& Cx) \)

6.3. Alternative Formulation of No, None, Nobody, etc. Instead of translating these expressions as existentials, you can express them equivalently as universals.

“None of Carrot-top’s jokes are funny,” translates as “Everything is such that (if it’s a Carrot-top joke, then it’s not funny).”
\( Jx = x \) is a Carrot-top joke.
\( Fx = x \) is funny.
\( \forall x (Jx \supset \neg Fx) \)
It is logically equivalent to \( \neg \exists x (Jx \& Fx) \).

“No one likes you, and nobody wants to be seen around you,” translates as “Everything is such that (if it is a person, then it does not like you), and everything is such that (if it is a person, then it does not want to be seen around you).”
\( Px = x \) is a person.
\( Lx = x \) likes you.
\( Wx = x \) wants to be seen around you.
\( \forall x (Px \supset \neg Lx) \& \forall x (Px \supset \neg Wx) \)
It is logically equivalent to \( \forall x (Px \supset (\neg Lx \& \neg Wx)) \) and \( \neg \exists x (Px \& (Lx \& Wx)) \).

6.4. Not All vs. Not Any. Be careful about where you put the negation.

“No all of the people in this room are rude.”
\( \neg \forall x (Px \supset Rx) \)

“No any of the people in this room are rude,” translates as “Not some of the people in this room are rude.”
\( \neg \exists x (Px \& Rx) \)

6.5. Articles.

“The squirrel is in the attic.”
\( Ax = x \) is in the attic
\( Sx = x \) is a squirrel
\( As \)
“The squirrels are playing.”
This is ambiguous between ‘most of the squirrels are playing’, and ‘all the squirrels are playing’. It also seems to imply plurality, which we can’t do. The first possibility we can’t do yet. The second possibility is \( \forall x (Sx \supset Px) \)

“A squirrel is in the attic.”
This is ambiguous between ‘there is exactly one squirrel in the attic’, and ‘there is at least one squirrel in the attic’. The first possibility we can’t do yet. The second possibility is \( \exists x (Sx \& Ax) \)

“A squirrel is a distant relative of the groundhog,” translates as “Squirrels are distant relatives of the groundhog.” \( \forall x (Sx \supset Rx) \)

“Squirrels are in the attic,” translates as “There are squirrels in the attic.” This is ambiguous between “There is more than one squirrel in the attic,” and “There is at least one squirrel in the attic.” \( \exists x (Sx \supset \exists x) \)

“Squirrels are mammals,” translates as “All squirrels are mammals.”
\( \forall x (Sx \supset Mx) \)

“Squirrels build nests in trees,” means, “By and large, squirrels build nests in trees.” We can’t translate this with predicates.

6.6. Only. Remember how ‘A if B’ is equivalent to ‘B only if A’? ‘Only’ sentences are universals where the direction of the horseshoe is flipped (the antecedent and consequent are exchanged).

“Only members are allowed in the clubhouse,” translates as “Every thing is such that (if it is allowed in the clubhouse, then it is a member).”
\( Cx = x \) is allowed in the clubhouse.
\( Mx = x \) is a member.
\( \forall x (Cx \supset Mx) \)

Notice how with the ‘only’ the subject is to the right of the horseshoe, and the verb phrase is to the left.

“Only the courageous can play,” translates as “Every thing is such that (if it can play, then it is courageous).”
\( Cx = x \) is courageous.
\( Px = x \) can play.
\( \forall x (Px \supset Cx) \)

How do we translate, “If only members of our club were invited, then all our members came”?\)
\( Mx = x \) is a member of our club.
\( Ix = x \) was invited.
\( Cx = x \) came.
\( \forall x (Ix \supset Mx) \supset \forall x (Mx \supset Cx) \)

“The only children who are allowed in this playground are those from Rosewood Elementary,” translates as “Only children from Rosewood Elementary are allowed in this playground.”
7. SUMMARY OF RULES FOR PREDICATE TRANSLATION

\[ Cx = x \] is a child.
\[ Ax = x \] is allowed to be in this playground.
\[ Rx = x \] is from Rosewood Elementary.
\[ \forall x((Cx \& Ax) \supset Rx) \]

Note that the translation is not \[ \forall x(Ax \supset (Cx \& Rx)) \] which is different in that it says adults are not allowed on the playground (and dogs are not allowed, and bananas are not allowed, and purple bow-ties are not allowed, etc.) Effectively, it says the only things allowed in the playground are Rosewood children. Notice how (compared to the universal claim “All children from Rosewood Elementary are allowed in this playground,” which is \[ \forall x((Cx \& Rx) \supset Ax) \]) we don’t strictly have a switch of the consequent with the antecedent but only with one conjunct in the antecedent. To see what’s going on, it is useful to use the so-called importation and exportation rules. Start with “All children from Rosewood Elementary are allowed in this playground,” which is \[ \forall x((Cx \& Rx) \supset Ax) \]. This is equivalent to \[ \forall x(Cx \supset (Rx \supset Ax)) \] which says that if you are a child, then if you are from Rosewood, you are allowed. Adding the ‘only’ to the sentence (to give us “Only children from Rosewood Elementary are allowed in this playground,”) makes us switch the Ax and Rx, without touching the Cx. This gives us \[ \forall x(Cx \supset (Ax \supset Rx)) \] which says that if you are a child, then if you are allowed, then you are from Rosewood. This is equivalent to the correct answer above, \[ \forall x((Cx \& Ax) \supset Rx) \], which says that all allowed children are from Rosewood. The reason we know we should not switch the Cx along with the Rx is that in the original sentence, it says “The only children who are allowed...” which means, “Among children, the only who are allowed...” It does not mean that the only things allowed in the playground are Rosewood children. Toys are allowed, slides, parents, etc.

7. Summary of Rules for Predicate Translation

\[ \forall \] is for ALL, every, each. ‘\[ \forall \]’ uses ‘\[ \supset \]’ to connect the subject restrictions with the verb phrase. For example, “Every gray, furry, mangy cat is old or weak or sick,” is \[ \forall x((Gx \& Fx \& Mx \& Cx)) \supset (Ox \lor (Wx \lor Sx))) \].

\[ \exists \] is for EXISTS, some, a, an. ‘\[ \exists \]’ uses ‘\[ \& \]’ to connect the subject restrictions with the verb phrase. For example, “Some gray, furry, mangy cat is old or weak or sick,” is \[ \exists x((Gx \& Fx \& Mx \& Cx)) \& (Ox \lor (Wx \lor Sx))) \].

- “Some S’s are A” is translated with \[ \exists x(Sx \& Ax) \].
- “No S’s are A” is translated as \[ \neg \exists x(Sx \& Ax) \].
- “All S’s are A” is translated as \[ \forall x(Sx \supset Ax) \].
- “Every S is an A” is translated as \[ \forall x(Sx \supset Ax) \].
- “Not all S’s are A” is translated as \[ \neg \forall x(Sx \supset Ax) \].
- “Not every S is an A” is translated as \[ \neg \forall x(Sx \supset Ax) \].
- Anytime you are using a word like ‘someone’ or ‘somebody’ or ‘anyone’ or ‘anybody’ or ‘everyone’ or ‘everybody’, there is an implicit restriction to people. Thus you must use a predicate that represents personhood like \[ Px \] to mean \[ x \] is a person.
- ‘Any’ could mean ‘some’ or ‘every’. Plug in both and see which sounds better. Then use the rules for the sentence that sounds better. The times when it translates as ‘some’ are when it is in the antecedent of a conditional, and a few rare occasions when it is negated.
• If there is an ambiguity in the sentence, and only one disambiguation can be translated into predicate logic, pick the disambiguation that can be translated into predicate logic. E.g., “Unicorns exist,” should be translated as “There exists at least one unicorn.”

• Indefinite articles are sometimes $\exists$, and other times $\forall$. To know which, think about whether the sentence applies to one specific object or to objects generally.

• ‘Only’ is translated with $\forall$, but with the usual order of the subject and predicate reversed. Be careful to keep any scope restrictors as part of the antecedent.

• Sentences that have individuals as their subjects don’t have quantifiers around the subject. Just put the predicate with the lowercase letter.

• If you have a conditional that draws a general conclusion about something, the use of ‘something’ is translated with a $\forall$. “If something is red, then it is square,” translates as $\forall x (Rx \supset Sx)$.

• If you have a conditional where neither antecedent nor consequent refer to anything in the other, translate with the ‘$\supset$’ as the main connective. “If something is red, then something is square,” translates as $\exists x Rx \supset \exists x Sx$.

8. Vacuous Quantification
CHAPTER 5

Relations

Relations are the name for connections between individuals. They are different from properties only in that properties apply to just a single individual. Relations apply to more than one.

Properties apply to a single individual:
- “The swing set is broken.”

Relations apply to two or more individuals:
- “Randy is in front of the bar.” (2 individuals)
- “The coffee table is between the sofa and the TV.” (3 individuals)
- “The goat ate the flower in my garden at o’clock.” (4 individuals)

1. Translating Relations

We represent relations with a capital letter and a superscript representing how many things they connect. We use variables $x$, $y$, and $z$ to tell us which individuals have which role in the English sentence.

$F_{xy} = x$ is a friend to $y$.

$k = \text{Katy}$

$m = \text{Michelle}$

$c = \text{Cliff}$

“Katy is a friend to Michelle.”

$F_{km}$

“Cliff is a friend to Katy.”

$F_{ck}$

“Cliff and Michelle are friends.”

$F_{cm} \& F_{mc}$.

$B_{xyz} = x$ is between $y$ and $z$.

$j = \text{Jill}$

$h = \text{the house}$

$c = \text{the car}$

$b = \text{Jill’s brother}$

“Jill is between her brother and the car.”

$B_{jbc}$

“Jill’s brother is between the house and the car.”

$B_{bhc}$

“The car is between Jill and her brother.”

$B_{cjb}$
“The car is between Jill’s brother and Jill.”

\( Bchj \)

(You should not use additional facts that you know about the ‘between’ relation to reorder the individuals. As far as logic goes, the last two logic sentences have different meanings even though the English versions are equivalent.)

### 1.1. Using Quantifiers with Relations.

We can quantify over relations, but we have to use a variable as a placeholder so that we know which individual in the relation is quantified.

- \( Oxy = x \text{ is older than } y \)
- \( m = \text{Michelle} \)
- \( c = \text{Michelle’s car} \)

“Michelle’s car is older than she is.”

\( Ocm \)

“Michelle is older than something.”

\( \exists xOm x \)

“Something is older than Michelle.”

\( \exists xOxm \)

“Everything is older than Michelle’s car.”

\( \forall xOxc \)

“Michelle is older than everything.”

\( \forall xOmx \)

“Nothing is older than Michelle.”

\( \neg \exists xOxm \)

“Something is older than everything.”

\( \exists x\forall yOxy \)

“Everything is older than something.”

\( \forall x\exists yOxy \)

\( Px = x \text{ is a person.} \)

\( Lxy = x \text{ likes } y. \)

\( b = \text{Beth} \)

“Everyone likes Beth.”

\( \forall y(Py \supset Lyb) \)

“Not everyone likes Beth.”

\( \neg \forall y(Py \supset Lyb) \)

“Something likes Beth.”

\( \exists zLzb \)

“Someone likes Beth.”

\( \exists z(Pz\&Lzb) \)

“Beth likes something.”

\( \exists xLbx \)

“Beth likes someone.”
2. Translating Multiply Nested Quantified Relations

\[ \exists x (P_x \& L_b x) \]

\[ P_x = x \text{ is a person.} \]

\[ L_{xy} = x \text{ likes } y. \]

\[ s = \text{Steve} \]

\[ w = \text{Wendy} \]

“If Steve likes someone then Wendy likes someone.”

\[ \exists z (P_z \& L_z s) \supset \exists y (P_y \& L_w y) \]

“Unless Wendy likes Steve, Wendy doesn’t like anyone.”

\[ \neg L_{ws} \supset \neg \exists y (P_y \& L_w y) \]

“Either Steve likes someone or Wendy likes everyone.”

\[ \exists z (P_z \& L_z s) \lor \forall y (P_y \supset L_w y) \]

“Everyone that Steve likes, Wendy likes too.”

\[ \forall y ((P_y \& L_s y) \supset L_w y) \]

“Anyone that likes Steve, is liked by Wendy and Steve both.”

\[ \forall y ((P_y \& L_y s) \supset (L_w y \& L_s y)) \]

“If no one likes Wendy, then Wendy doesn’t like everyone.”

\[ \neg \exists z (P_z \& L_z w) \supset \neg \forall y (P_y \supset L_w y) \]

“Wendy likes everyone who likes her in return.”

\[ \forall y ((P_y \& L_y w) \supset L_w y) \]

“Wendy and Steve like the same people.”

\[ \forall y ((P_y \& L_y w) \supset L_s y) \land \forall y ((P_y \& L_s y) \supset L_w y) \]

\[ \forall y (P_y \supset ((L_w y \supset L_s y) \land (L_s y \supset L_w y))) \]

“Everyone likes someone.”

\[ \forall x (P_x \supset \exists y (P_y \& L_x y)) \]

2. Translating Multiply Nested Quantified Relations

\[ K_{xy} = x \text{ knows } y. \]

\[ P_x = x \text{ is a person.} \]

\[ E_x = x \text{ has an education.} \]

“Everyone who has an education is known to someone who knows Paul.”

Stage I: Write down the main verb of the sentence as the main relation. For the occupants of the relation, insert the appropriate predicates and relations, whether simple or complex, marking the related individuals with the appropriate quantifiers. Put the subscripts in the order that makes sense given the meaning of the relation. Use different variables for each subscript.

\[ K_{xy} (P_x \& K_x y), \forall y (P_y \& E_y) \]

Stage II: Pull the quantifiers from the subscript position out in front of the relation, using the appropriate connective, \( \supset \) for \( \forall \) and \& for \( \exists \). Do this in the order given by the English sentence. (The rule about English order does not always work, but is a surprisingly good rule of thumb. As before, ‘only’ can throw a wrench into the machinery being described here.)

\[ \forall y ((P_y \& E_y) \supset K_{xy} (P_x \& K_x y), y) \]

which then becomes
∀y((Py&Ey) ⊃ ∃x((Px&Kxp)&Kxy))
and because we have no more subscripts for K we are done.

Rxy = x is richer than y.
Px = x is a person.
Ox = x owns a car.
“Anyone who owns a car is richer than anyone who doesn’t.”
“Everyone who owns a car is richer than everyone who doesn’t.”

∀x((Px&Ox) ⊃ ∃y((Py&¬Oy) ⊃ Rxy))
becomes
∀x((Px&Ox) ⊃ ∀y((Py&¬Oy) ⊃ Rxy))

Notice that because the lexical order (the order of the English words) tells you
which quantifier to pull out first, switching a sentence from active voice to passive
voice, changes the meaning of the statement.

Cxy = x cares for y.
Vx = x is a veterinarian.
Px = x is a professional.
Fx = x has fleas.
Tx = x has ticks.
Dx = x is a dog.

“This means, “Every dog that has fleas or ticks is cared for by some professional veterinarian.”

∀y((Dy&(Fy ∨ Ty)) ⊃ Cxy)
∀y((Dy&(Fy ∨ Ty)) ⊃ ∃x((Vx&Px)&Cxy))

This means, “Every dog that has fleas or ticks is such that there is some veterinarian
who is professional and cares for that dog.”

Compare the following: “Some professional veterinarian cares for every dog
with fleas or ticks.”

Cxy (Vx&Px) ∃y(Dy&Fy) ∃z(Dz&Tz)

This means, “Some veterinarian who is professional cares for every dog that has
fleas or ticks.”

Bxyz = x sits between y and z.
Px = x is a person.
Sx = x snores.
Cx = x coughs.

“No one who snores sits between someone who coughs and someone who doesn’t
cough.”

“There does not exist someone who snores and sits between someone who coughs
and someone who doesn’t cough.”

Bxyz = x sits between y and z.
Px = x is a person.
Sx = x snores.
Cx = x coughs.

“No one who snores sits between someone who coughs and someone who doesn’t
cough.”

“There does not exist someone who snores and sits between someone who coughs
and someone who doesn’t cough.”

Bxyz = x sits between y and z.
Px = x is a person.
Sx = x snores.
Cx = x coughs.
becomes
\[\neg \exists x((P_x \& S_x) \& B_x, \exists y((P_y \& C_y) \& \exists z((P_z \& \neg C_z))))
\neg \exists x((P_x \& S_x) \& \exists y((P_y \& C_y) \& \exists z((P_z \& \neg C_z))))
\neg \exists x((P_x \& S_x) \& \exists y((P_y \& C_y) \& \exists z((P_z \& \neg C_z)) \& B_{xyz}))
\]

\[G_{xy} = x \text{ gives to } y.
B_{xy} = x \text{ is better than } y.
P_x = x \text{ is a person.}
\]
“Anyone who gives to someone is better than everyone who doesn’t give to anyone.”
“Every person who gives to some person is better than every person who doesn’t give to some person.”
\[B_x(P_x \& G_x \exists z P_z), \forall y(P_y \& \neg G_y \exists z P_z)
\forall x((P_x \& G_x \exists z P_z) \supset B_x, \forall y(P_y \& \neg G_y \exists z P_z))
\forall x((P_x \& G_x \exists z P_z) \supset \forall y((P_y \& \neg G_y \exists z P_z) \supset B_{xy}))
\forall x((P_x \& \exists z(P_z \& G_{xz})) \supset \forall y((P_y \& \neg \exists z(P_z \& G_{yz}) \supset B_{xy})))
\]
CHAPTER 6

Identity

The resources we have in our logic do not allow us to express identity, that an object under one name and an object under another name can be said to be the same object. To see why our logic is incapable of expressing identity, let’s explore three different strategies for expressing identity using properties, and we will see how each strategy has inadequacies. Let’s try to translate, “The chancellor is Elaine Coleman.”

In the first strategy, we symbolize a property of being Elaine Coleman.

c = the chancellor
Ex = x is Elaine Coleman.

The sentence translates as Ec. One problem with this way of symbolizing identity is that

\( \text{The chancellor is Elaine Coleman.} \)
\( \text{Thus, Elaine Coleman is not the chancellor’s administrative assistant.} \)

looks valid, but

\( Ec \quad \neg Ea \)

is invalid.

In the second strategy, we symbolize a property of being the chancellor.

Cx = x is the chancellor.
e = Elaine Coleman

The sentence translates as Ce. One problem with this way of symbolizing identity is that

\( \text{The chancellor is Elaine Coleman.} \)
\( \text{Thus, Harvey Kemp is not the chancellor.} \)

looks valid, but

\( Ce \quad \neg Ch \)

is invalid.

In the third strategy, we symbolize both properties.

Cx = x is the chancellor.
Ex = x is Elaine Coleman.

The sentence translates as \( \forall x((Cx \supset Ex) \& (Ex \supset Cx)) \& \exists x Ex \)

One problem with this way of symbolizing identity is that

\( \text{The chancellor is Elaine Coleman.} \)
\( \text{Thus, no one other than Elaine Coleman is the chancellor.} \)
6. Identity

looks valid, but we have no way to translate the conclusion.

What is missing in our translation is the ability to capture the difference between identity and predication. Consider different uses of the verb ‘to be’:

- Elaine is the chancellor.
- Elaine is tired.
- Elaine is my mother.
- Elaine is overwhelmed by my project.
- Elaine is blonde.
- Elaine is the president of the FRP Corporation.

In some cases, we are using ‘is’ to say that Elaine is identical with some individual. In other cases, we are using ‘is’ to say that Elaine has some property or quality or relation. In order to strengthen our language so that it can make the distinction, we introduce a new symbol ‘=’, and some new rules.

First, we introduce one new rule into our definition of wff’s.

- Any capital letter is a wff.
- Any capital letter immediately followed by n lowercase letters (either individuals or variables) is a wff.
- If \( \alpha \) is a wff, then \( \neg \alpha \) is a wff.
- If \( \alpha \) is a wff, and \( \beta \) is a wff, then \( \alpha \lor \beta \) is a wff.
- If \( \alpha \) is a wff, and \( \beta \) is a wff, then \( \alpha \supset \beta \) is a wff.
- If \( \alpha \) is a wff and \( x \) is a variable, then \( \forall x \alpha \) is a wff.
- If \( \alpha \) is a wff and \( x \) is a variable, then \( \exists x \alpha \) is a wff.
- \( (a = b) \) is a wff if \( a \) and \( b \) are variables or constants.
- Nothing else is a wff.

We don’t need to modify our definition of a statement as a wff with no free variables. In addition, we can abbreviate anything of the form \( \neg (a = b) \) as \( a \neq b \). Also, to cut down on the number of parentheses, you can drop the parentheses around an equality.

1. Truth Trees with Identity

To address our new symbol ‘=’ we need two new rules.

- Substitution of Identicals (=): From any sentence of the form \( a = b \) and another sentence that includes \( a \), derive a sentence that is made by replacing as many instances of \( a \) as you like with \( b \). (Or replace as many \( b \’\)s as you like with \( a \).) Cross out the sentence that you replaced.
- Any instance of \( (a \neq a) \) counts as an explicit contradiction, where \( a \) is any term.

Example 5. Substitution of Identicals

1. \( \exists x (Kx \& (Sx \lor (Mc \& Rx_c))) \) \hspace{1cm} Premise
2. \( c = f \) \hspace{1cm} Premise
3. \( \exists x (Kx \& (Sx \lor (Mc \& Rx_f))) \) \hspace{1cm} 1, 2, =

Example 6. Identity
2. Translations with Identity

2.1. Exceptions.

\( Ux = x \) is upset.
\( Px = x \) is a person.
\( d = \text{Dirk} \)

“Someone other than Dirk is upset.”
There exists a person who isn’t Dirk and who is upset.
\( \exists x((Px \& x \neq d) \& Ux) \)

“Everyone is upset except possibly Dirk.”
Every person who isn’t Dirk is upset.
\( \forall x((Px \& x \neq d) \supset Ux) \)

“Everyone except Dirk is upset.”
Every person who isn’t Dirk is upset.
\( \forall x((Px \& x \neq d) \supset Ux) \)

“Everyone besides Dirk is upset.”
Every person who isn’t Dirk is upset.
\( \forall x((Px \& x \neq d) \supset Ux) \)

“Everyone is upset except Dirk.”
Every person who isn’t Dirk is upset, and Dirk is upset.
\( \forall x((Px \& x \neq d) \supset Ux) \& \neg Ud \)

“Dirk is the only person who is upset.”
Dirk is upset, and every person who isn’t Dirk is not upset.
\( Ud \& \forall x((Px \& x \neq d) \supset \neg Ux) \)

“Only Dirk is upset.”
Dirk is upset, and every person who isn’t Dirk is not upset.
\( Ud \& \forall x((Px \& x \neq d) \supset \neg Ux) \)

2.2. Superlatives.

\( Txy = x \) is taller than \( y \).
\( Mx = x \) is a mountain.
\( e = \text{Mount Everest} \)
\( k = \text{K2} \)
"Mount Everest is the tallest mountain."
Mount Everest is a mountain, and every mountain except Mount Everest is such that Mount Everest is taller than it.

\[\text{Me} \& \forall x ((\text{M}x \& x \neq e) \implies Te)\]

"Mount Everest is the next tallest mountain after K2."
Mount Everest is a mountain, and K2 is a mountain, and K2 is taller than Mount Everest, and every mountain that isn’t K2 and isn’t Mount Everest is such that Mount Everest is taller than it.

\[(\text{Me} \& \text{Mk}) \& \forall x ((\text{M}x \& x \neq k) \implies Tkx) \& \forall x ((\text{M}x \& x \neq e \& x \neq k) \implies Te)\]

2.3. Counting. The identity relation allows us to symbolize quantitative features. Specifically, it allows us to express the concepts of 'more than', 'fewer than', 'an equal number of', and to include specific numbers in these relations to express claims like, “There are exactly five birds flying,” and “There are no more than three people in the room.”

To express "There are at least \(n\) objects with the properties \(P\), \(Q\), and \(R\)," the pattern is to set up \(n\) variables under an existential quantifier and then inside the scope of the quantifier to state (1) that each of the \(n\) variables has the properties \(P\), \(Q\), and \(R\), and (2) that none of the variables is equal to any other. One must conjoin every pair of quantified variables in an inequality.

\[Ax = x\] was arrested.
\[Px = x\] is a person.

"At least one person was arrested."
There exists an \(x\) who is a person and arrested.

\[\exists x (Px \& Ax)\]

"At least two people were arrested."
There exists an \(x\) and a \(y\) where \(x\) is a person and arrested; and \(y\) is a person and arrested and \(y\) not the same as \(x\).

\[\exists xy (Px \& Ax \& Py \& Ay \& y \neq x)\]

"At least three people were arrested."
There exists an \(x\) and a \(y\) and a \(z\) where \(x\) is a person and arrested; and \(y\) is a person and arrested and is not the same as \(x\); and there exists a \(z\) who is a person and is arrested and is not the same as \(x\) and is not the same as \(y\).

\[\exists xyz (Px \& Ax \& Py \& Ay \& Pz \& Az \& y \neq x \& z \neq x \& z \neq y)\]

"At least four people were arrested."
There exists an \(x\) who is a person and is arrested; and there exists a \(y\) who is a person and is arrested and is not the same as \(x\); and there exists a \(z\) who is a person and is arrested and is not the same as \(x\) and is not the same as \(y\); and there exists a \(w\) who is a person and is arrested and is not the same as \(x\) and is not the same as \(y\) and is not the same as \(z\);
To express “There are at most \(n\) objects with the properties \(P, Q,\) and \(R\),” the pattern is to set up \(n + 1\) variables under a universal quantifier and then inside the scope of the quantifier to state a conditional with (1) the antecedent that each of the \(n + 1\) variables has the properties \(P, Q,\) and \(R\), and (2) the consequent that (at least) two of the variables are equal to each other. One must disjoin every pair of quantified variables in an inequality in the consequent.

“At most one person was arrested.”
For every person \(x\) and every person \(y\), if \(x\) is a person and arrested and \(y\) is a person and arrested, then they are the same person.
\[
\forall xy ((P_x \& A_x \& P_y \& A_y) \supset x = y)
\]

“At most two people were arrested.”
For every person \(x\) and every person \(y\) and every person \(z\), if \(x\) is a person and arrested and \(y\) is a person and arrested and \(z\) is a person and arrested, then either \(x\) is the same as \(y\) or \(y\) is the same as \(z\).
\[
\forall xyz ((P_x \& A_x \& P_y \& A_y \& P_z \& A_z) \supset (x = y \vee x = z \vee y = z))
\]

“At most three people were arrested.”
For every person \(x\) and every person \(y\) and every person \(z\) and every person \(w\), if \(x\) is a person and arrested and \(y\) is a person and arrested and \(z\) is a person and arrested and \(w\) is a person and arrested, then either \(x\) is the same as \(y\) or \(y\) is the same as \(z\) or \(z\) is the same as \(w\) or \(w\) is the same as \(x\).
\[
\forall xyz w ((P_x \& A_x \& P_y \& A_y \& P_z \& A_z \& P_w \& A_w) \supset (y = x \vee z = x \vee z = y \vee w = x \vee w = y \vee w = z))
\]

To express “There are exactly \(n\) objects with the properties \(P, Q,\) and \(R\),” one just conjoins the expressions for “there are at least \(n\)...” and “there are at most \(n\)...”

“Exactly one person was arrested.”
At least one person was arrested and at most one person was arrested.
\[
\exists x ((P_x \& A_x) \& \forall xy ((P_x \& A_x \& P_y \& A_y) \supset x = y)
\]

“Exactly two people were arrested.”
At least two people were arrested and at most two people were arrested.
\[
\exists xy ((P_x \& A_x \& P_y \& A_y \& y \neq x) \& \forall xyz ((P_x \& A_x \& P_y \& A_y \& P_z \& A_z) \supset (x = y \vee x = z \vee y = z))
\]
Although we have the resources to make quantitative statements, we don’t yet have the ability to do arithmetic. Arithmetic requires our adding functions to our logic.

2.4. Infinity. Now that we can express the concept of equality in our logic, we can create a set that is true in all infinite models and false in all finite models. This is just the set that contains all the expressions of the form “There are at least $n$ things.” The set would look like

$$\{\exists x (y \neq x), \exists y (y \neq x \& z \neq x \& z \neq y), \exists y (x \neq y \& z \neq x \& z \neq y \& w \neq x \& w \neq y \& w \neq z), \text{ etc.}\}$$

While we do have a set that is true iff the model is infinite, we cannot yet express with a single sentence in our logic the idea that there are an infinity of things, nor can we express the idea that there are a finite number of things. This requires what is called second order logic.

3. Implicit Properties of Binary Relations

Some concepts have built in qualities that our quantifier logic with identity can uncover. For example, the argument

$K2$ is taller than Mount Everest.
Thus, Mount Everest is not taller than $K2$.

is valid. But when we translate it as

$$Tke \quad \neg Tke$$

we get an invalid argument. The reason is that the validity of the natural language version hinges on the meaning of ‘tall’ which our straightforward translation does not capture. Luckily in this case, the relevant aspect of the tallness relation can be expressed in our logic. We need to express the idea that tallness is anti-symmetric, that if $x$ is taller than $y$, then $y$ is not taller than $x$.

Here are some premises that are implicit in the meaning of ‘tall’:

- $\forall x \neg Txx$ (Nothing is taller than itself.)
- $\forall xy (Txy \supset \neg Tyx)$ (If object $x$ is taller than $y$, then $y$ is not taller than $x$.)

When judging arguments using the concept ‘tall’ for validity, these statements should be added to the premises because they are implicitly assumed by any use of the word ‘tall’. Our argument now becomes

$$Tke \quad \forall x \neg Txx \quad \forall xy (Txy \supset \neg Tyx)$$

$$\neg Tke$$

which, as you can check, is valid.

You do not need to make the implicit facts about tallness explicit in your translation of any single sentence that uses the word ‘tall’, but you do need to ensure that your translation is consistent with these. That is why it is important to translate “Mount Everest is the tallest mountain” as “Every mountain except Mount Everest is such that Mount Everest is taller than it.”
5. FUNCTIONS

There are implicit premises that recur. First, a relation can be reflexive, meaning that every object bears that relation to itself. For example, the ‘is the same age as’ relation is reflexive because every object is the same age as itself. For another example, the ‘is at least as big as’ relation is reflexive because every object is at least as big as itself. If we have a relation $K$ that is reflexive, we can make the reflexivity explicit by adding $\forall x Kxx$ as a premise. Some relations are irreflexive, meaning that every object does not bear that relation to itself. For example, the ‘is a brother of’ relation is irreflexive because no one is his own brother. To express that $K$ is an irreflexive relation, we say $\forall x \neg Kxx$.

4. Presupposition

Let $Bx = x$ is bald, and consider “The present king of France is bald.” If we assume there is an individual $k$ that is the present king of France, we can translate it as $Bk$.

But we might also try to model kingship as a predicate. Let $Kx = x$ is a king of France. Now consider how to translate the sentence.

$\exists x ((Kx \& Bx) \& \forall y (Ky \supset x = y))$

5. Functions

Sometimes we have a need to translate statements like the following:

“There is a student whose father is rich.”

One way to translate this is the following:

$\exists x (Sx \& \exists y (Ry \& Fyx))$ where $Sx$ means $x$ is a student, $Ry$ means $y$ is rich, and $Fyx$ means $y$ is the father of $x$.

This translation does not capture the fact that there is only one father. We could add a further premise to clarify that every student has a father who has existed at some time: $\forall x (Sx \supset \exists y Fyx \& \forall z (Fzx \supset z = y))$ We won’t be concerned with taking account of the temporal component, that a father may no longer exist at all times when the student exists.

Because we frequently encounter circumstances where such logical connections are needed, it is convenient to introduce some new structure in our logical formalism to make these situations easier to write down.

We introduce the idea of an $n$-place function $f(x_1, x_2, x_3, \ldots, x_n)$. A function plays the role of an individual in our logic. It can be located anywhere a variable or constant goes.

There are two implicit constraints on any function $f(x_1, x_2, x_3, \ldots, x_n)$.

Existence: $\forall x_1 \forall x_2 \forall x_3 \ldots \forall x_n (\exists y (y = f(x_1, x_2, x_3, \ldots, x_n)))$

Uniqueness: $\forall x_1 \forall x_2 \forall x_3 \ldots \forall x_n \forall y_1 \forall y_2 ((y_1 = f(x_1, x_2, x_3, \ldots, x_n) \& y_2 = f(x_1, x_2, x_3, \ldots, x_n)) \supset y_1 = y_2)$

First order logic is the full quantifier logic with identity and with function symbols.
CHAPTER 7

Applications

1. Anselm’s Ontological Argument

Ontological arguments for the existence of God attempt to establish the existence of God from purely a priori premises, especially premises about the concept of God.

A

In David Lewis’ article, “Anselm and Actuality,” (http://www.jstor.org/stable/2214320) he tries to formalize one of Anselm’s ontological arguments for the existence of God.

Saint Anselm’s Ontological Argument:

(1) Whatever exists in the understanding can be conceived to exist in reality.
(2) Whatever exists in the understanding would be greater if it existed in reality than if it did not.
(3) Something exists in the understanding, than which nothing greater can be conceived.
(4) Thus, something exists in reality, than which nothing greater can be conceived.

1.1. Translation of the argument. Premise 1: Whatever exists in the understanding can be conceived to exist in reality.

$Ux = x$ is understandable.
$Wx = x$ is a world.
$Exw = x$ exists in $w$.

All understandable beings have some conceivable world where they exist.

$\forall x (Ux \supset \exists w (Ww & Exw))$

Premise 2: “Whatever exists in the understanding would be greater if it existed in reality than if it did not.”

$Ux = x$ is understandable.
$Wx = x$ is a world.
$Exw = x$ exists in $w$.
$Gxyzw = \text{the greatness of } x \text{ in } y \text{ exceeds the greatness of } z \text{ in } w$

For any understandable being $x$, and for any worlds $y$ and $z$, if $x$ exists in $y$ but does not exist in $z$, then the greatness of $x$ in $y$ exceeds the greatness of $x$ in $z$.

$\forall xyz ((Ux \& Wy \& Wz \& Exy \& \neg Ez) \supset Gxyz)$

Premise 3: “Something exists in the understanding, than which nothing greater can be conceived,” which translates as, “There is an understandable being whose greatness cannot be conceived to be exceeded by the greatness of anything,” which
translates as, “The greatness of $x$ is not exceeded by the greatness in any conceivable world of any being $y$.” The remaining question about this translation is, “Which greatness of $x$?” There are four different versions of premise 3 that Lewis considers:

Premise 3a: (The actual greatness of $x$) There is an understandable being $x$, such that for no world $w$ and being $y$ does the greatness of $y$ in $w$ exceed the greatness of $x$ in the actual world.

Let $a = $ the actual world.  

$\exists x(\forall x \& \neg \exists w \exists y(Ww \& Gywx))$

Premise 3b: (The greatest greatness of $x$) There is an understandable being $x$ and world $z$, such that for no world $w$ and being $y$ does the greatness of $y$ in $w$ exceed the greatness of $x$ in $z$.

$\exists x \exists z(Ux \& Wz \& \neg \exists w \exists y(Ww \& Gywxz))$

Premise 3c. (Any greatness of $x$) There is an understandable being $x$, such that for no worlds $z$ and $w$ and being $y$ does the greatness of $y$ in $w$ exceed the greatness of $x$ in $z$.

$\exists x(\forall x \& \neg \exists z \exists w \exists y(Ww \& Wz \& Gywxz))$

Premise 3d: (The greatness of $x$ in the same world as the greatness of the being who is compared to $x$) There is an understandable being $x$ such that for no worlds $w$ and being $y$ does the greatness of $y$ in $w$ exceed the greatness of $x$ in $w$.

$\exists x(\forall x \& \neg \exists z \exists w \exists y(Ww \& Gywxw))$

Conclusion: Something exists in reality, than which nothing greater can be conceived.

$\exists x(Exa \& \neg \exists w \exists y(Ww \& Gywx))$

1.2. Evaluation of Validity. Which disambiguations of Anselm’s argument are valid? The versions with premises 3a and 3c are valid. What’s more 3c implies 3a, so you can prove that the argument with 3a is valid and then prove that 3c implies 3a, and that will ensure the argument version with 3c is valid.

The versions with premises 3b and 3d are invalid. Thus, we can set aside consideration of 3b and 3d as irrelevant because the corresponding arguments are invalid.

1.3. Evaluation of the Premises. To analyze the reasonability of Anselm’s ontological argument, we need to discuss how plausible the premises are. Refer to Lewis’s article to read his discussion of 3c. It is a bit more subtle than we can get into right now. So why should we believe 3a: “There is an understandable being $x$, such that for no world $w$ and being $y$ does the greatness of $y$ in $w$ exceed the greatness of $x$ in the actual world”? In other words, why is the actual world the home of the greatest greatness?

If 3a is plausible in and of itself, that’s because there is something special about our world, the actual world. One possibility is that the actual world is special because of some particular facts about our world that make this so. This isn’t terribly plausible though, since we can imagine the actual world being better in almost every respect. The second possibility is that the actual world is special
because there is something special about being actual. But why would actuality be a special (greatness-making property)?

Many of the features of our language about reality seem to treat reality as an indexical. An indexical expression is an expression whose denotation always varies according to the context of utterance. Examples include personal pronouns, demonstratives, ‘here’ and ‘now’.

- ‘I’ refers to the whoever the speaker is.
- ‘Here’ refers to the spatial location of the speaker when he utters ‘here’.
- ‘Now’ refers to the moment when the speaker utters ‘now’.
- ‘Actual’ refers to the world of the speaker.

Lewis wants us to believe that alternate universes are every bit as real as our universe. Being real or actual is not a special property. Santa Claus and transparent metal exist in some worlds. They just don’t inhabit the same universe that you and I inhabit. In brief, reality is an indexical notion. Relative to the world I inhabit, you exist. Relative to the world you and I exist in, Santa Claus, doesn’t exist. There is an alternate universe where Santa Claus is thinking to himself, “I exist” and Santa is right. He exists in the world where he says “I exist.” But none of us exist in that world. We are mere possibilities to Santa just as Santa is to us.

The conclusion Lewis then wants us to draw is that we don’t have any special property of actuality, and thus no reason to believe in 3a, and thus no reason to be motivated by Anselm’s argument.

Lewis’ views on actuality are not widely shared. One reason for doubting them is that our world is special in many interesting ways that would be inexplicable without there being something special about the world we inhabit. For example, many of the regularities of our world can be described in a remarkably simple mathematical form. it is compatible with all we know that all the regularities of nature as we know them will cease operation tomorrow. However, it routinely happens that the simple regularities continue to hold, and the specialness of our universe (the specialness of its laws of nature) seem to be the best explanation for the enduring regularities.

2. Models and Theories

A theory is a set of sentences. A theory in propositional logic is something like \{R, W & E, \neg (R \lor (E \supset \neg W))\}. A theory in quantifier logic is something like \{U1d & \forall x ((P1x & \neg (x = d)) \supset \neg U1x), \exists x (P1x & A1x), R3haf\}.

These theories are just strings of grammatically organized symbols. They are just symbols that fit the rules for sentences (wffs with only bound variables).

The theories only have meaning when they are given an interpretation, when we settle on how the symbols relate to something.

For natural languages like English, we can think of interpretations as mappings from sentences to sets of possible worlds. For example, the sentence “No pigeons are red.” is mapped to a set containing all the possible worlds that completely lack red pigeons.

For propositional logic, we think of interpretations as mappings from sentences to either TRUE or FALSE. For example, one possible interpretation of the theory \{Q\} is to map the sentence Q to FALSE. Another interpretation is to map Q to TRUE. These are the only possible interpretations for the theory \{Q\}. There
are constraints on how we can interpret the symbols. For example if we map $Q$ to FALSE and $E$ to TRUE, then we have to map $\neg(E \lor \neg Q)$ to FALSE. The constraints on interpretations in propositional logic are given by the truth tables for $\neg, \&$, $\lor$, and $\supset$. We use truth tables to tell us what all the possible interpretations of a theory are. For example, the theory $\{F\&R, \neg(M \supset \neg M), \neg R\}$ has 4 possible interpretations, one where $M$ and $R$ map to TRUE, one where $M$ and $R$ map to FALSE, one where $M$ maps to TRUE and $R$ maps to FALSE, and one where $R$ maps to TRUE and $M$ maps to FALSE.

For quantifier logic, we can think of interpretations as mappings from sentences to sets of possible models, where the models contain objects that may have properties and relations. For example, we will discuss models with objects like $d$, $h$, $k$, with properties ‘spherical’, ‘cubic’, ‘yellow’, and ‘green’, and with the relations ‘above’, and ‘below’.

For the illustrations, we will stick to interpretations where the following predicates and relations are kept fixed:

- $Sx = x$ is spherical.
- $Cx = x$ is cubic.
- $Yx = x$ is yellow.
- $Gx = x$ is green.
- $Axy = x$ is above $y$.
- $Bxy = x$ is below $y$.

Here are some sentences that are true of the model pictured in figure 1:

$Gk$
$Ck$\&$Cq$
$\forall x ((Gx \lor Cx) \supset Axh)$
$\neg \exists x (Yx \& \exists y (Gy \& Ayx))$

Here are some sentences that are true of the model pictured in figure 2:

$\neg Bdh$
$\forall x (Yx \& \neg Bxd)$
$\forall x (Gx \lor Axh)$
$\neg \exists x (Yx \& Axh)$

Which of the following sentences are true of the model pictured in figure 3?
Suppose we add a relation $R_{xy}$. Figure 4 has an arrow going from $x$ to $y$ if and only if the relation $R$ holds between $x$ and $y$. Here are some sentences that are true of the model pictured above:

This relation pictured in figure 5 is interesting because it expresses a reflexive relation. A relation is reflexive if and only if every object bears that relation to itself, i.e., $\forall x R_{xx}$

\[
\begin{align*}
\exists x (\neg Cx \& \neg B_k x) \\
\forall x (Y x \supset B_k x) \\
G_d \lor \neg \forall x (C x \supset \neg A x d) \\
\neg \exists x (S x \& \exists y A x y)
\end{align*}
\]
This relation pictured in figure 6 has no objects that bear the relation to itself. We call this kind of relation irreflexive. A relation is reflexive if and only if every object does not bear that relation to itself, i.e., $\forall x \neg Rxx$. A relation can be reflexive or irreflexive or neither.

The model in figure 7 is neither reflexive (because $\neg Rqq$) nor irreflexive (because $Rdd$). This model does show a symmetric relation. A relation is symmetric if and only if whenever one object bears the relation to another, the other bears it back to the original, i.e., $\forall x (Rxy \supset Ryx)$.

Relations like the one pictured in figure 8 do not have any objects that bear a relation to some object and have that same object bear the relation back to it. Relations like these are called asymmetric. The defining property of an asymmetric relation is $\forall x (Rxy \supset \neg Ryx)$. A relation is either symmetric, asymmetric or neither.

Figure 8 illustrates a relation that is transitive. A relation is transitive if and only if whenever there is a chain of relations between one object and another, there is a (direct) relation between that object and the other, i.e., $\forall xyz ((Rxy \& Ryz) \supset Rxz)$.

When a relation is reflexive, symmetric and transitive, we say that it is an equivalence relation. An equivalence relation is shown in figure 9.

**Exercise 17.** Draw a model with a yellow cube $s$, and three spheres including $k$. Have the model make the following sentences true: $\{Rss, \forall x (x \neq s \supset \neg Rxx), \exists y (Sy \& Ryk), \exists z (Gz \& \forall y (Sy \supset (Rzy \& Ryz)))\}$
Exercise 18. Draw a model with a yellow cube \( r \), and three spheres. Have the model make the following sentences true: \( \{ \forall x (\neg Rsx \& \neg Rsx), \exists y (Gy \& \forall x (x \neq r \supset Rx), \exists y \exists z (Ryz \& Rzy) \} \)

2.1. Rules for Reading Model Pictures. Truth
- If there is an arrow going from \( b \) to \( c \), then \( Rbc \) is true.
- If there is not an arrow going from \( b \) to \( c \), then \( Rbc \) is false.

Reflexivity
- If every object has a self-loop, the relation is reflexive.
- If no object has a self-loop, the relation is irreflexive.
- If some have a self-loop and others don’t, it is neither.

Symmetry
- If every time there is an arrow from \( b \) to \( c \), there is also an arrow going from \( c \) to \( b \), the relation is symmetric.
- If every time there is an arrow from \( b \) to \( c \), there is not an arrow going from \( c \) to \( b \) and there are no loops, the relation is anti-symmetric.
- If (some pairs have arrows going back and forth or there are loops and others don’t have arrows going back and forth), it is neither.
- (Self-loops don’t affect symmetry. Ignore them.)
- (Self-loops do affect anti-symmetry. If there is a self-loop, the relation is not anti-symmetric.)
- (Objects without arrows don’t affect symmetry. Ignore them.)

Transitivity
- If every time there is an arrow from \( b \) to \( c \), and an arrow going from \( c \) to \( f \), there is also an arrow going directly from \( b \) to \( f \), the relation is transitive.
- If every time there is an arrow from \( b \) to \( c \), and an arrow going from \( c \) to \( f \), there is not an arrow going directly from \( b \) to \( f \), the relation is intransitive.
- If some triplets of objects fit the pattern and others don’t, it is neither.
- (Objects without arrows don’t affect transitivity. Ignore them.)
- (To be transitive, a pair of objects with arrows going back and forth have to also have self-loops on both objects.)
- (To be intransitive, there can’t be any self-loops.)

2.2. Natural Relations. How do we check for reflexivity, symmetry and transitivity for relations that are not pictured?

For reflexivity, ask “Do the objects bear that relation to themselves?” If they have to, it is reflexive. If they can’t, it is irreflexive. If they can but don’t have to, it is neither. Examples:
- \( x \) is a cousin to \( y \). Irreflexive
- \( x \) is similar to \( y \). Reflexive
- \( x \) is bigger than \( y \). Irreflexive
- \( x \) is a friend to \( y \). Irreflexive
- \( x \) is located within 12 feet of \( y \). Reflexive

For symmetry, ask “If one object bears that relation to another, does the other bear it back in return?” If it has to, it is symmetric. If it can’t, it is anti-symmetric. If it can but doesn’t have to, it is neither. Examples:
- \( x \) is a cousin to \( y \). Symmetric
- \( x \) is similar to \( y \). Symmetric
• $x$ is bigger than $y$. Anti-symmetric
• $x$ is a friend to $y$.
• $x$ is located within 12 feet of $y$. Symmetric

For transitivity, ask “If the first object bears that relation to a second, and the second bears it to a third, does the first bear it to the third? If it has to, it is transitive. If it can’t (and it is irreflexive), it is intransitive. If it can but doesn’t have to, it is neither. Examples:

• $x$ is a cousin to $y$.
• $x$ is similar to $y$.
• $x$ is bigger than $y$. Transitive
• $x$ is a friend to $y$.
• $x$ is located within 12 feet of $y$.

3. Elucidating Concepts

6 cheerleaders have formed what is called a pyramid, (really a triangle). Jodie is above Angela, and Conrad is below Angela. Can we infer that Conrad is below someone who is above Angela?

$A_j \land B_c \\
\exists x (P_x \land (A_x \land B_c x))$

This is not valid. We have to add additional implicit premises to give our logic a chance to show that it is a valid inference. First we need to add the seemingly obvious, that Jodie, Angela and Conrad are all people.

$P_j \\
P_a \\
P_c \\
A_j \\
B_c \land (A_j \land B_c x))$

We also write down a complete set of sentences that express as much as we can express about aboveness relation, $A$, and the belowness relation, $B$.

Here is what we can say about $A$:

• It is irreflexive: $\forall x \neg A_x x$
• It is asymmetric: $\forall xy (A_x y \supset \neg A y x)$
• It is transitive: $\forall xyz ((A_x y z \supset A x z z)$

Here is what we can say about $B$:

• It is irreflexive: $\forall x \neg B_x x$
• It is asymmetric: $\forall xy (B_x y \supset \neg B y x)$
• It is transitive: $\forall xyz ((B_x y z \supset B z z)$

The following principle captures all we know about their connection.

$\forall x \forall y ((A_x y \supset B y x) \& (B_x y \supset A y x))$

The final argument becomes

$P_j \\
P_a \\
P_c \\
A_j$
4. FAMILY RELATIONS

In the Computer Science department, you can learn about some programming languages called Prolog, or A-Prolog or some variant. What these languages let you do is to enter in some information in the form of sentences of first-order logic (what we have been learning all semester), and then the computer does all the inferences (truth trees). This can be used as a tool to explore creating artificial intelligence. We humans start off a computer agent with some general facts, and then the computer is able to make deductions.

Suppose we try to teach the computer about families. We want to put in some general principles about mothers, sisters, cousins, etc.

Let’s start out delineating the concept of a brother. What concepts can we use to break down the meaning of being a brother?

(1) To be a brother you have to be male.
(2) To be a brother you have to have the same parents.

Let’s start out assuming we have a property of being male and relationship of parent.

Let Pxy be x is a parent of y.
Let Mx be x is male.
Let Bxy be x is a brother to y.

We want to express, “x is a brother to y if and only if x is male and x has two parents w and z, and y has the same two parents w and z.”

We code this up as a general statement ranging over all people x and y, ∀x∀y((Mx&∃z∀w((Pzx&Pzy)&(Pwz&Pwy))) ≡ Bxy).

After putting this information into the computer, we need to check to see whether it works. Suppose we tell the computer the following additional facts:

- Dustin is a parent to Gunner. Pdg
- Rachel is a parent to Gunner. Prg
- Dustin is a parent to Brytni. Pdb
- Rachel is a parent to Brytni. Prb
- Gunner is male. Mg
- Brytni is female. Fb
- Dustin is male. Md
- Rachel is female. Fr

When we ask the computer, “Is Gunner a brother of Brytni?” we are asking whether Bgb follows logically from the premises above. The computer answers “Yes.”
When we ask the computer, “Is Brytni a brother of Dustin?” \((Bbd?\)\), the computer answers that it doesn’t know. This is because it cannot infer that Brytni is, and it cannot infer that she isn’t. It isn’t (so far as we have programmed it) able to tell that Brytni is not male, even though it knows that Brytni is female.

We need some principle that tells us that a person can’t be both male and female. This brings up some interesting questions of sexuality. Do we want to say everyone is either male or female? Do we want to say that no one can be both? For simplicity assume that there is no one of dual sex. We do this by adding \(\forall x \neg (Mx \& Fx)\) where \(Fx\) means \(x\) is female. Now, when we ask the computer, “Is Brytni a brother to Dustin?” \(Bbd?\) the computer will correctly answer “No.”

Suppose we tell the computer some new facts:

1. Austin is male. \(Ma\)
2. Dustin is a parent to Austin. \(Pda\)

Now, when we ask the computer, “Is Gunner a brother to Austin?” \(Bga\) The computer answers “Yes,” but it ought to answer “I don’t know,” because they might only be half-brothers.

This is because \(\forall x \forall y ((Mx \& \exists z \exists w ((Pzx \& Pzy) \& (Pwx \& Pwy))) \equiv Bxy)\) is poorly stated. It doesn’t take account of the fact that the \(z\) and \(w\) need to be different people. Moreover, we need one to be the mother and one to be the father. We need to tell the computer what a mother is: \(\forall x \forall y (Mxy \equiv (Fx \& Pxy))\) and what a father is: \(\forall x \forall y (Fxy \equiv (Mx \& Pxy))\)

Then our definition of brotherhood becomes:

\[\forall x \forall y ((Mx \& \exists z \exists w ((Fzx \& Fzy) \& (Mwx \& Mwy))) \equiv Bxy)\]

Now when we ask, “Is Gunner a brother to Austin?” the computer will answer, “I don’t know,” because it doesn’t know that Rachel is Austin’s mother.

Let’s go ahead and tell it that Rachel is Austin’s mother. \(Mra\) So now it will answer yes to “Is Gunner a brother to Austin?” and “Is Austin a brother to Gunner?”

Suppose we test it further by asking, “Who are Brytni’s brothers?” The computer responds, “Gunner and Austin.” Note that the computer is telling us the information so far as it knows. The computer is not telling us that those are the only people that are Brytni’s brothers, but just that those are the only brothers the computer knows about.

Suppose we test it further by asking, “Who are Gunner’s brothers?” The computer responds, “Gunner and Austin.” But wait! It should have just said Austin. Why did it say Gunner was one of Gunner’s brothers? Because we didn’t tell the computer that a person can’t be his own brother. So let’s add it in as follows: \(\forall x \neg Bxx\).

Now when we ask, “Who are Gunner’s brothers?” the computer answers, “Gunner, Austin, Rachel, Dustin, Brytni.” How did that happen? It happened because we got a contradiction in our premises. The definition of brother that we had still implies that Gunner is his own brother, and our new premise implies that he isn’t. Thus, we have given the computer an inconsistent set of information, and from an inconsistent set, every statement whatsoever follows.

Instead of saying just \(\forall x \neg Bxx\), we need to modify the definition of brotherhood as well. \(\forall x \forall y ((Mx \& \neg (x = y) \& \exists z \exists w ((Fzx \& Fwx) \& (Mzy \& Mwy))) \equiv Bxy)\) With this change, the computer will correctly answer, “Dustin.”
Suppose we ask “Is Gunner a father to Brytni?” The computer answers, “I don’t know.” But we know they are brother and sister, so they can’t be father and daughter.

Before we add a rule ruling out siblings from being parent and child, let’s test it some more. Suppose we add the fact that Phil is male. Then we ask, “Is Phil a father to Brytni?” The computer answers, “I don’t know.” But in this case too, we know that the answer is “No.” Because we know that a person only has one father and one mother. If we can get the computer to solve this problem we can get it to fix the problem where it doesn’t know that Gunner is not the father to Brytni.

So we just need to state the idea that a person only has one father. Actually, while we’re at it, we might as well add something else. That everyone has a one father. Our statement then is that everyone has exactly one father:

\[ \forall x (\exists z Fzx \& \forall y \forall z ((Fxz \& Fyz) \supset x = y)) \] and the same for mother:

\[ \forall x (\exists z Mzx \& \forall y \forall z ((Mxz \& Myz) \supset x = y)) \]

Let’s summarize the relations and predicates we have used so far:

- Let \( Pxy \) be \( x \) is a parent of \( y \).
- Let \( Mx \) be \( x \) is male.
- Let \( Fx \) be \( x \) is female.
- Let \( Mxy \) be \( x \) is a mother of \( y \).
- Let \( Fxy \) be \( x \) is a father of \( y \).
- Let \( Bxy \) be \( x \) is a brother to \( y \).

Let’s summarize the general facts we have in the computer:

\[ \forall x \forall y (Mxy(Fx \& Pxy)) \]
\[ \forall x \forall y (Fxy(Mx \& Pxy)) \]
\[ \forall x \forall y ((Mx \& \neg (x = y)) \& \exists z \exists w ((Fxz \& Fzy) \& (Mwx \& Myw))) \equiv Bxy \]
\[ \forall x (\exists z Mzx \& \forall y \forall z ((Mxz \& Myz) \supset x = y)) \]
\[ \forall x (\exists z Fxz \& \forall y \forall z ((Fxz \& Fyz) \supset x = y)) \]
\[ \forall x Pxx \]

Let’s summarize the particular facts we have in the computer:

- Dustin is a parent to Gunner. \( Pdg \)
- Rachel is a parent to Gunner. \( Prg \)
- Dustin is a parent to Brytni. \( Pdb \)
- Rachel is a parent to Brytni. \( Prb \)
- Dustin is a parent to Austin. \( Pda \)
- Rachel is a parent to Austin. \( Pra \)
- Gunner is male. \( Mg \)
- Brytni is female. \( Fb \)
- Dustin is male. \( Md \)
- Rachel is female. \( Fr \)
- Austin is male. \( Ma \)
- Phil is male. \( Mp \)

**Exercise 19.** How do you define a sister relation? A grandmother relation? A cousin relation?

Let’s summarize the additional relations and predicates:

- Let \( Pxy \) be \( x \) is a parent of \( y \).
- Let \( Sxy \) be \( x \) is a sister to \( y \).
- Let \( Bxy \) be \( x \) is a brother to \( y \).
• Let $S_{xy}$ be $x$ is a sister to $y$.
• Let $G_{xy}$ be $x$ is a grandparent to $y$.
• Let $G^f_{xy}$ be $x$ is a grandfather to $y$.
• Let $G^m_{xy}$ be $x$ is a grandmother to $y$.
• Let $C_{xy}$ be $x$ is a cousin to $y$.
• Let $A_{xy}$ be $x$ is an aunt to $y$.
• Let $U_{xy}$ be $x$ is an uncle to $y$.
• Let $J_{xy}$ be $x$ is a nephew to $y$.
• Let $N_{xy}$ be $x$ is a niece to $y$.
• Let $O_{xy}$ be $x$ is a great-grandparent to $y$.

(Treat expressions with superscripts, e.g. $G^f$, just like a relation letter. This allows us to more easily remember what the relation symbols stand for.)

Let's summarize the general facts we already have in the computer:

- $\forall xy (M_{xy} \equiv (Fx \& Py))$ definition of mother
- $\forall xy (F_{xy} \equiv (Mx \& Px))$ definition of father
- $\forall xy ((Mx \& \neg(x = y)) \& \exists z (Fzx \& Fzy) \& \exists w (Mwx \& Mwy)) \equiv B_{xy}$ definition of brother
- $\forall xy ((Fx \& \neg(x = y)) \& \exists z (Fzx \& Fzy) \& \exists w (Mwx \& Mwy)) \equiv S_{xy}$ definition of sister
- $\forall x \neg B_{xx}$ No one is his or her own brother.
- $\forall x \neg S_{xx}$ No one is his or her own sister.
- $\forall x \exists z (Mxz \& \forall yz ((Mxz \& Myz) \supset x = y))$ Everyone has exactly 1 mother
- $\forall x \exists z (Fxz \& \forall yz ((Fxz \& Fyz) \supset x = y))$ Everyone has exactly 1 father
- $\forall x \neg (Mx \& Fx)$ No one is both male and female.
- $\forall x (Mx \lor Fx)$ Everyone is either male or female.
- $\forall x P_{xx}$ No one is his or her own parent.

How do you tell the computer enough so that it can answer questions about grandmothers and grandfathers?

- $\forall xy (G_{xy} \equiv \exists z (Pxz \& Pzy))$ definition of grandparent
- $\forall xy (G^m_{xy} \equiv (Gxy \& Fx))$ definition of grandmother
- $\forall xy (G^f_{xy} \equiv (Gxy \& Mx))$ definition of grandfather
- $\forall x \neg G_{xx}$ no one is his or her own grandparent

How do you tell the computer enough so that it can answer questions about cousins?

- $\forall x \forall y (C_{xy} \equiv (\exists z (G^m_{xz} \& G^m_{zy}) \& \exists w (G^f_{wx} \& G^f_{wy}) \& (x \neq y)) \& \neg (B_{xy} \lor S_{xy}))$ definition of cousin

How do you tell the computer enough so that it can answer questions about half-sisters and half-brothers?

- $\forall x \forall y ((Mx \& (\exists z (Fzx \& Fwx) \equiv \neg \exists w (Mzy \& Mwy)))) \equiv B^{1/2}_{xy}$ definition of half-brother
- $\forall x \forall y ((Fx \& (\exists z (Fzx \& Fwx) \equiv \neg \exists w (Mzy \& Mwy)))) \equiv S^{1/2}_{xy}$ definition of half-sister

4.1. Ancestorhood. A question that turns out to be surprisingly interesting is, “Can the concept ‘ancestor’ be properly defined in terms of first-order logic and the parent relation. It is easy to state the proper inference from relations of parenthood to relations of ancestorhood. We just note that being a parent is a
special case of being an ancestor:

\[ \forall xy (Pxy \supset Axy) \]

where \( Pxy \) represents that \( x \) is a parent of \( y \) and \( Axy \) represents that \( x \) is an ancestor of \( y \).

Then, we can impose a transitivity requirement on \( A \) in order to ensure that chains of parenthood result in direct relations of ancestorhood:

\[ \forall xyz ((Axy \& Ayz) \supset Axz) \]

That information suffices to connect parenthood relations to ancestor relations, but what about the other way around? One thing we might want to say is that no one is their own ancestor. If there is time travel to the past, this may not be true, but otherwise it seems save to say this. We can simply rule out self-ancestorhood with an additional premise about the meaning of ‘ancestor’:

\[ \forall \neg Axx \]

This suffices to rule out any looped paths of parenthood.

Another fact we know about ancestor relations is that when \( x \) is an ancestor of \( y \), there must be a chain of parent relations that lead from \( x \) to \( y \). The problem is, “How do we express (in first-order logic) that a chain must exist?” We could claim the following:

\[ \forall xy (Axy \supset (Pxy \lor \exists z(Axz \& Azy))) \]

This says that whenever \( x \) is an ancestor of \( y \), either \( x \) is a parent of \( y \) or there is someone intermediate between \( x \) and \( y \) ancestorwise.

We can go further and also state that

\[ \forall xy (Axy \supset (Pxy \lor \exists z(Pxz \& Azy))) \]

and that

\[ \forall xy (Axy \supset (Pxy \lor \exists z(Axz \& Pzy))) \]

The first of these states that whenever \( x \) is an ancestor of \( y \), either \( x \) is a parent of \( y \) or \( x \) has a child who is an ancestor of \( y \). The first of these states that whenever \( x \) is an ancestor of \( y \), either \( x \) is a parent of \( y \) or \( y \) has a parent who is a descendant of \( y \). Together, they ensure that whenever \( x \) is an ancestor of \( y \), there is a chain of parenthood relations leading from \( x \) down to \( y \) as well as a chain of parenthood relations leading from \( y \) up to \( x \). However, it does not rule out the following possibility

\[ y \rightarrow y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow \cdots \text{gap} \cdots \rightarrow x_3 \rightarrow x_2 \rightarrow x_1 \rightarrow x \]

where there are an infinite number of \( y_i \) and an infinite number of \( x_i \), such that the parenthood relations never bridge the gap between the \( y_i \)'s and the \( x_i \)'s but where the ancestorhood relations do bridge the gap. That is, every \( x_i \) is an ancestor of every \( y_i \).

So, the problem is that all the axioms we have stated for the meaning of ancestorhood are compatible with the above model, yet we know that the ancestor relation requires a linked chain of parenthood relations. The only way we can rule out the above model is to rule out the existence of a parenthood gap. One way to do this would be to rule out an infinite chain of intermediates, but there is no way to express the concept ‘finite’ in first-order logic.
The Meta-Theory of First-Order Logic

In this chapter, we will explore several important properties of first-order logic, and compare first-order logic to several other weaker and stronger logics.

In particular, we will find that first-order logic is complete but undecidable.

A logic is decidable iff there exists a computer algorithm that can take any statement (or theory) $\alpha$ and correctly assess (in a finite amount of time) the status of $\alpha$ as consistent or inconsistent.

A logic is complete iff there exists a computer algorithm that can take any statement (or theory) $\alpha$ and correctly assess (in a finite amount of time) that $\alpha$ is inconsistent when it is inconsistent.

The difference between these two is that decidability is logically stronger. That means any logic that is decidable is also complete, but a logic that is complete might not be decidable. The way that a logic can be complete but undecidable is that there are computer programs such that when an inconsistent $\alpha$ is input, the program always eventually detects that it is inconsistent, but when a consistent $\alpha$ is input, there are two possibilities: (1) the program might detect that $\alpha$ is consistent and (2) the program might keep running forever in an infinite loop without ever making a decision about whether $\alpha$ is consistent.

So, in first-order logic, you can write a program that will take any inconsistent statement, and it will correctly identify it as inconsistent, but no matter how you program the computer, if it always correctly identifies the inconsistent statements and never misidentifies the consistent statements, there will always be some consistent statements that lead to the program never producing an answer.

1. Decidability

This section summarizes the important lessons from Chapter 6 of Richard Jeffrey’s textbook *Formal Logic: Its Scope and Limits*.

The main point of the chapter is to prove that first-order logic is undecidable. (Look at the previous section if you are unsure what that means.)

The proof of undecidability has the following structure:

If first-order logic is decidable, then the halting problem can be solved.

The halting problem cannot be solved.

Thus, first-order logic is undecidable.

The premises are supported with their own arguments as follows.

1.1. Defense of “The halting problem cannot be solved.”. An abacus computer program is said to *halt* if and only if it completes running after any finite number of steps.
A *halting program* is an abacus computer program that accurately evaluates whether some chosen abacus computer program will eventually halt. Any halting program accepts as input the ordered sequence, \((m, x_1, x_2, x_3, \ldots)\) where \(m\) represents some abacus computer program and \((x_1, x_2, x_3, \ldots)\) represents the initial state of that abacus computer when program \(m\) is run. The halting program by definition always produces an output that represents the correct answer to the question, “Would abacus computer program \(m\) halt if it were started with initial state \((x_1, x_2, x_3, \ldots)\)?” (A halting program never gets stuck in an infinite loop.)

In order for the halting program to “know” which abacus computer programs is represented by the input \(m\), we need to have some way of labeling all abacus computer programs. So we need some list where every possible abacus computer program appears eventually somewhere in the list. It does not matter exactly what order they are in. We know such a list can be constructed because we can list all programs that have one node, then all programs that have two nodes, then all programs that have three nodes, and so on. For any given number of nodes there are only a finite number of distinct abacus computer programs. Thus, every program will appear somewhere on the list.

The *halting problem* is the task of finding a halting program that will correctly evaluate all possible abacus computer programs \(m\) for all possible inputs \((x_1, x_2, x_3, \ldots)\).

Here are two proofs that the halting problem cannot be solved.

1.1.1. First Proof. The first proof is of the form

*If the halting problem could be solved, then there would exist a self-contradictory abacus computer program.*

*A self-contradictory abacus computer program cannot exist.*

*Thus, the halting problem cannot be solved.*

A *self-contradictory* abacus computer program is one whose description is inconsistent with the constraints that all abacus computer programs must follow.

Let’s start by specifying the (hypothetically assumed) halting program \(H\). We can assume the special case of an \(H\) that takes just two numbers as input. The first register contains \(m\) pebbles which represents the label for the abacus computer program that \(H\) is going to evaluate. The second register contains \(n\) pebbles representing the number of pebbles that will be located in the first register when program \(m\) is run. We may stipulate that \(H\) outputs zero pebbles in register one whenever \(H\) decides that if program \(m\) were run with \(n\) pebbles in its first register, it would eventually halt. \(H\) outputs one pebble in register one whenever \(H\) decides that if program \(m\) were run with \(n\) pebbles in its first register, it would keep running forever.
To construct the self-contradictory abacus computer program, we link together three components, as depicted in Fig. 1. The first component merely copies register one into register two. The second component is just the (hypothetically assumed) halting program. The third component is designed to instigate the contradiction by halting if there is a pebble in register one and not halting if there is no pebble in register one.

This abacus computer program has some location \( m \) in the complete list of all abacus computer programs. It is self-contradictory in the special case where its input in register one is \( m \). The contradiction arises in that case because (1) the sub-program \( H \) must produce no pebbles if and only if the program \( m \) as a whole halts, and (2) the bilking sub-program is constructed so that the machine \( m \) as a whole halts if and only if the sub-program \( H \) produces a pebble. Thus, when the abacus program receives its own label as input, it halts if and only if it does not halt. Thus, it cannot work properly for all possible inputs. Thus, it is a self-contradictory abacus computer program. Because there is nothing about the copy component and the bilking component that violates the rules of a well-behaved abacus computer, the fault must like in the \( H \) component. Thus, \( H \) is guaranteed to fail for at least some inputs, no matter how it is constructed. Thus, the halting problem is impossible to solve.

1.1.2. Second Proof. The second proof uses a busy beaver program. A busy beaver program is an abacus computer program that evaluates the running time of other abacus computer programs. The running time of an abacus computer program \( X \) is the number of computational steps \( X \) takes when started with all registers empty in order to get to a stage where it halts with all its registers empty. If an abacus computer program does not halt, or if it halts with pebbles in its registers, then its running time is zero, representing that it is disqualified. The point of a busy beaver program is to hold a competition among abacus computer programs in order to see which ones have the longest running time. It helps to imagine a bunch of programs trying to run as long as they can but without running infinitely long.

Specifically, a busy beaver program takes \( n \) pebbles in register one as input and produces an output in register two equal to the longest running time of any abacus computer program that has exactly \( n \) nodes. Fig. 2 depicts five of the many two-node programs. The longest running of the five is the one that runs for three computational steps. There happens to be no other two-node program with a better score, so the busy beaver program is required to output 3 pebbles in register two.

Let \( r(n) \) be the longest running time of any abacus computer program that has exactly \( n \) nodes. A busy beaver program can be defined as any program that calculates \( r(n) \) for any whole number \( n \). The busy beaver problem is the task of
finding a busy beaver program. In other words, the busy beaver problem is the task of determining whether \( r(n) \) is computable.

The second proof is of the form

*If the halting problem could be solved, then the busy beaver problem can be solved.*

*The busy beaver problem can be solved.*

*The halting problem cannot be solved.*

The first premise can be defended as follows. If the halting problem could be solved, then you could calculate \( r(n) \) by going through the list of \( n \)-node programs one at a time, checking to see if they halt, and then simulate the ones that halt to measure their running times. With there being only a finite number of distinct \( n \)-node programs, all of the programs will eventually be checked, and the answer is just the maximum running time found.

The second premise can be proven as follows. First, let us assume for the sake of discussion that there is a busy beaver program, BB. All abacus computer programs have a finite number of nodes, so let \( k \) represent the number of nodes in BB. Figure 3 illustrates a program that uses register two as a counter so that BB will run twice. (We can assume without loss of generality that BB is programmed to avoid using register two.) The first time BB runs, its input is \( n \) and so it will output \( r(n) \) in register one. Then, when the program follows the loop back to BB, it will run a second time with input \( r(n) \) and so it will output \( r(r(n)) \). So, the program as a whole will halt with \( r(r(n)) \) in register one. Because the pebbles in register one can only get there one increment at a time, the program must execute a 1+ node at least \( r(r(n)) \) times. Because it executes other commands as well, the program as a whole must take more than \( r(r(n)) \) steps. Thus, we have an example of a program with \( n + k + 2 \) nodes takes more than \( r(r(n)) \) steps. Thus, \( r(n + k + 2) > r(r(n)) \).

It should be obvious that \( r(n) \) always increases as \( n \) increases because the possibility of an extra node allows you in a worst case scenario to make a program run one step longer. (Just add a 1- node at the start of one of the longest running \( n \)-node programs. That will increase the running time by one unit while doing nothing to shorten the running time of the existing program.) Thus, in general, if \( r(x) > r(y) \), then \( x > y \). Thus, in particular \( n + k + 2 > r(n) \).

Figure 3 illustrates that in general, \( r(n + 1) \geq 2n + 1 \) because there will always be this program that increments register one \( n \) times, decrements it \( n \) times, and then exits.

Combining the two inequalities results in, \( n + 1 + k + 2 > r(n + 1) \geq 2n + 1 \). Thus, \( n + k + 3 > 2n + 1 \). Thus, \( k > n - 2 \). Because this equation needs to hold true for all \( n \) in order for BB to count as a successful busy beaver program, \( k \) cannot be a finite number. But \( k \) must be a finite number because it is the number of nodes
in an abacus computer program. Thus, the assumption that BB is a bona fide busy beaver program has led to a contradiction. Thus, BB cannot exist. Because no special constraints were imposed on BB, its failure to exist can only be a result of its needing to satisfy the definition of a busy beaver program. Thus, there can be no busy beaver program.

Having demonstrated the truth of both premises, the conclusion validly follows that the halting problem cannot be solved.

1.2. Defense of “If first-order logic is decidable, then the halting problem can be solved.”. The main idea behind the proof for this claim is that the logical structure of any abacus computer program can be expressed in first-order logic. We can also express the claim that an abacus machine has halted. Then, we will examine whether the premises that express the structure of the abacus computer program entails that the computer will halt.

The scheme for doing this is to have one variable \( t \) represent the time starting with time zero, have a bunch of individuals represent each arrow in the abacus computer program, and have a finite number of variables representing the contents of any registers that the program uses.

I will just illustrate with the abacus computer program depicted in Fig. 5. Because these programs might require more constants than we have available, we will be using a new set of constants that start with the constant \( 0 \), and then continue to the constant \( 1 \), which represents \( 0' \), the successor of zero. The constant \( 2 \), which represents \( 1' \), the successor of one. This successor function is a standard one-place function. The only special properties we need it to obey is that \( 0 \) is not the successor of anything, and that any given successor is the successor of exactly one thing. That just ensures that the sequence of successors generated by starting with zero, continues on towards infinity without ever looping around.

For each arrow that leads to a + command, there is a single premise.

The 1+ node expresses the idea “If at time \( t \), the computer is at arrow 1, with register one being \( x \) and register two being \( y \), then at time \( t' \) (which represents the next unit of time), the computer is at arrow 2, with register one being \( x' \) (which means \( x + 1 \)) and register two being \( y \). We let a four-place relation represent this claim:

\[
\forall txy(R_{\xi txy} \supset R_{\xi txy'})
\]

The 2+ node expresses the idea “If at time \( t \), the computer is at arrow 2, with register one being \( x \) and register two being \( y \), then at time \( t' \) (which represents the next unit of time), the computer is at arrow 3, with register one being \( x \) and register two being \( y \) (which means \( y + 1 \)).

\[
\forall txy(R_{\xi txy} \supset R_{\xi txy'})
\]
The 1- node expresses two ideas: “If at time \( t \), the computer is at arrow 3, with register one being \( 0 \) and register two being \( y \), then at time \( t' \) (which represents the next unit of time), the computer is at arrow 4, with register one being \( 0 \) and register two being \( y \). Also, “If at time \( t \), the computer is at arrow 3, with register one being a positive number \( x' \) and register two being \( y \), then at time \( t' \) (which represents the next unit of time), the computer is at arrow 6, with register one being \( x \) and register two being \( y \).

\[
\forall txy (R_{t30y} \supset R_{t40y})
\]

\[
\forall txy (R_{t3x'y} \supset R_{t6xy})
\]

Similarly, the 2- node expresses two ideas corresponding to the path from 4 to 5 and the path from 4 to 7.

\[
\forall txy (R_{t4x0} \supset R_{t7y0})
\]

\[
\forall txy (R_{t4xy'} \supset R_{t5xy})
\]

To complete the translation of the program into first-order logic, we need statements for the 6 to 3 path, and for the 7 to 5 and 7 to 7. These just repeat information that we have already written down. We just need to take the statement corresponding to the 2 to 3 path and replace references to 2 with 6. Similarly, we replace references to 4 with references to 7 in the two statements corresponding to the 2- node.

\[
\forall txy (R_{t6xy} \supset R_{t3xy'})
\]

\[
\forall txy (R_{t7x0} \supset R_{t7y0})
\]

\[
\forall txy (R_{t7xy'} \supset R_{t5xy})
\]

So now we have a full listing of the logical content of all the nodes. It should be obvious that a finite list of such information can be generated for any abacus computer program (which have finite nodes). These will serve as premises.

Another premise that we need is the information that at time zero, the computer starts at the entry arrow 1 with nothing in any of the registers. This is simply

\[
R_{01000}
\]

The conclusion of the argument is going to be the claim that the computer halts. To do this, we just say that the \( R \) relation holds at some time when we are
at one of the exit arrows. In this case, there is only one exit arrow, so the conclusion is

$$\exists t x y R_{t \sigma} x y$$

If there were more exit arrows, the conclusion would be a disjunction of claims like the one made for arrow 5.

Now, the argument associated with the program depicted in Fig. 5 has been specified. The premises include all the rules for all the nodes, and that the program starts at the entry arrow, 1. The conclusion is that there is some time at which the program reaches the exit arrow, 5.

The ultimate goal of this subsection is to show that if first-order logic is decidable, then the halting problem can be solved. We can prove this if we can prove that any program halts if and only if its associated argument is valid.

It is trivial that ‘associated argument is valid $$\supset$$ program halts’ because each of the lines in the argument was designed to represent the initial state and structure of the program, and so if it follows logically from that information that there is a time when the program reaches the exit arrow, the corresponding program must halt.

It is only a bit more difficult to show that ‘program halts $$\supset$$ associated argument is valid.’ One can do so by showing that the premises imply the existence of a unique state of the computer for each time. One of the premises already tells us what the initial state is. And all the other premises tell us how to get from a state at one time to a unique state at another time. Thus, for any time before the program halts, there is a unique state the computer is in. So if the computer halts at some time, it must have a description of the form $$R_{k \sigma} m n$$ where $$k$$, $$m$$, $$n$$ are constants. At that implies that the conclusion is true because it just states $$\exists t x y R_{t \sigma} x y$$.

So, if there were a decision procedure for figuring out the validity of any argument, then there would be a decision procedure for figuring out the validity of arguments that associated with some abacus computer program. Because such a decision procedure would determine which programs halt, it would solve the halting problem. Because that cannot be done, there must be no decision procedure for figuring out the validity of any argument.

2. Interpretations

To construct a model, we start with some base materials:

1. A domain of individuals. This is the set of objects that constitutes the universe.
2. Each constant gets mapped to one object in the universe.
3. Each n-place function symbol gets mapped to a map of n-tuples of objects to a single object.
4. Each proposition letter gets mapped either to TRUE or to FALSE.
5. Each n-place predicate gets mapped to n-tuples of objects. These intuitively represent the objects which bear the property or relation.

Then, all other expressions get their interpretation from the base materials according to the following rules.

1. The special relation ‘=’ gets mapped to the set containing all pairs of objects.
(2) Each function that has no variables in it gets mapped to the appropriate object.
(3) Each predicate letter $P$ that has no variables following it gets mapped to TRUE if the n-tuplet is in $P$ and FALSE otherwise.
(4) Each statement of the form $\neg \alpha$ gets mapped to TRUE if $\alpha$ gets mapped to FALSE, and FALSE otherwise. Notice that this rule does not apply to wffs in general, only to statements.
(5) Each statement of the form $\alpha \& \beta$ gets mapped to TRUE if $\alpha$ gets mapped to TRUE and $\beta$ gets mapped to TRUE, and FALSE otherwise. Notice that this rule does not apply to wffs in general, only to statements.
(6) Each statement of the form $\forall x \alpha[x]$ gets mapped to TRUE if there is a new name $n$ you can hypothetically introduce such that no matter what object $n$ gets mapped to, the corresponding interpretations of compound statements results in $\alpha[n]$ being mapped to TRUE. Otherwise, $\forall x \alpha[x]$ gets mapped to FALSE.
(7) Each statement of the form $\exists x \alpha[x]$ gets mapped to TRUE if there is a new name $n$ you can hypothetically introduce such that there is one object $n$ gets mapped to where the corresponding interpretations of compound statements results in $\alpha[n]$ being mapped to TRUE. Otherwise, $\exists x \alpha[x]$ gets mapped to FALSE.

3. Completeness

To say that the tree system that we have been using for first-order logic is complete is to say the following: Whenever the set of initial statements is inconsistent (unsatisfiable), every possible tree that follows from it will have all closed branches. An equivalent formulation applied to arguments is that whenever an argument is valid, there will exist a truth tree proof of it.

4. Soundness

To say that the tree system that we have been using for first-order logic is sound is to say the following: Whenever a truth tree has all closed branches, the set of initial statements is inconsistent (unsatisfiable). An equivalent formulation applied to arguments is that whenever a truth tree indicates that an argument is valid (by closing all branches), it is a valid argument. The equivalent (contrapositive) way to express soundness is to say that soundness holds when the following rule is obeyed: Whenever the set of initial statements is consistent (satisfiable), there will always be at least one open branch during any stage of the truth tree construction process.

Soundness is a property that every respectable proof system must obey. If a proof system didn’t obey soundness, then the proof system would “prove” that an argument was valid when it wasn’t. It would be awful to have a conceptual system where it makes sense to have a proof an invalid inference.
CHAPTER 9

Foundations of Mathematics

1. Infinity

A set $S$ is countable iff there exists some way of putting the elements of $S$ into a one-to-one correspondence with the natural numbers (or some subset of the natural numbers). That means, if $S$ is countable, then there is some formula or algorithm for putting the elements of $S$ in a sequence such that for every element, a person who is checking the list step by step at a constant rate will reach the element in a finite amount of time.

A set $S$ is uncountable iff $S$ is not countable. That is, if $S$ is uncountable, then there is no formula or algorithm for putting the elements of $S$ in a sequence such that for every element, a person who is checking the list step by step at a constant rate will reach the element in a finite amount of time. That is, if $S$ is uncountable, any list will omit some elements of $S$.

Examples of countable sets: natural numbers, integers, rational numbers.

Examples of uncountable sets: irrational numbers, real numbers, functions of integers.

1.1. Order Theory. The relation $R$ forms a strict partial order on set $S$ iff

- $\forall x Rxx$
- $\forall xy \neg (Rxy \& Ryx)$
- $\forall xyz ((Rxy \& Ryz) \supset Rxz)$

A strict partial order $R$ on $S$ forms a strict total order on set $S$ iff

- $\forall xy (Rxy \lor Ryx)$

A strict partial order $R$ on $S$ is dense iff

- $\forall xy (Rxy \supset \exists z (Rxz \& Rzy))$

Consider that the rational numbers and the irrational numbers both have a strict total order imposed on them by the ‘less than’ relation, $<$. They are also both dense with respect to $<$. Furthermore, neither of them are continuous subsets of the reals because any two distinct rational numbers has an irrational number in between them, and any two distinct irrational numbers has a rational number in between them. So, the fact that there are more irrationals than rationals shows that uncountability does not appear to be the result of the difference between a non-continuous set (like the rationals) and a continuous set (like the reals).

1.2. Expressing Finitude and Infinitude. There is no way to express the sentence “There exist only a finite number of things,” in first-order logic. That is, there is no single sentence of first-order logic that is true in all and only the finite models.
You can express the axiom of finitude in second-order logic as follows. The statement
\[ \forall R^2((\forall x Rxx \land \forall xy((Rxy \land Ryx) \supset Rxz)) \supset \exists y \forall x (x \neq y \supset Ryx) \]
is true in all and only finite models. Notice that what it says in effect is that no matter what strict partial ordering you place on the objects in the universe, there will always be a unique maximum object using that ordering. In any infinite universe, there will be some ways of imposing a strict ordering such that there is no maximum; any infinite list of objects will do.

There is also no way to express the sentence “There exist an infinite number of things,” in first-order logic. That is, there is no single sentence of first-order logic that is true in all and only the infinite models.

You can express infinitude in second-order logic by negating the axiom of finitude above.

And you can express an axiom of countability as follows:
\[ \exists z \exists s \forall X (Xz \land \forall x (Xx \supset Xs(x)) \supset \forall x Xx) \]
This states that there is some starting point, \( z \), which serves as a zero, and there is some function \( s(x) \), which serves as a successor function, such that for any monadic predicate that applies to zero and all its successors, it applies to everything. The axiom of countability is true in all and only the countable models, that is the finite models and countably infinite models.

But you can state a *theory* of first-order logic that is true in all and only the infinite models. That theory is just the infinite set that contains the formula for “There exists at least one object” and the formula for “There exists at least two objects” and so on for every finite number \( n \). The set as a whole can only be true for infinite models. Any finite model will make an infinite number of the statements in the theory false.

### 2. Robinson Arithmetic

We can express fundamental axioms of arithmetic with first-order logic.

Let \( 0 \) be the constant (which we will eventually interpret as the number zero) and let \( s(x) \) be a function which we call the successor of \( x \). We will interpret the successor of \( x \) as the number that follows \( x \) in counting the ordinary way. For example, 15 is the successor of 14. The successor function is often written with the “prime” symbol so that the successor of \( x \) is \( x' \).

We will also use two two-place functions, addition and multiplication. For example, the addition function is defined so that \( \text{sum}(x, y) \) outputs the sum of \( x \) and \( y \), and we often write this function as \( x + y \).

These are the axioms of \( Q \), Robinson Arithmetic:

1. \( \forall x \forall y (x \neq y \supset s(x) \neq s(y)) \)
2. \( \forall x (x \neq 0 \supset s(x)) \)
3. \( \forall x (x \neq 0 \supset \exists y (x = s(y))) \)
4. \( \forall x (x + 0 = x) \)
5. \( \forall x \forall y (x + s(y) = s(x + y)) \)
6. \( \forall x (x \times 0 = 0) \)
7. \( \forall x \forall y (x \times s(y) = (x \times y) + x) \)

With just these axioms, you can prove a lot of arithmetic.
3. Peano Arithmetic

One can augment the power of $Q$ by adding a (first-order) induction schema. Consider the countably infinite set of formulas of the form

$$\forall x_1, \ldots, x_n (\phi(o, x_1, \ldots, x_n) \land \exists y (\phi(y, x_1, \ldots, x_n) \supset \phi(s(y), x_1, \ldots, x_n))) \supset \forall y \phi(y, x_1, \ldots, x_n)$$

where $\phi(o, x_1, \ldots, x_n)$ is some statement of first-order logic. Each one of axioms says, if the statement $\phi$ holds for the base case (where $y$ equals zero) and it holds for the $y = n_1$ case whenever it holds for the $y = n$ case, then it holds for all cases. The result of adding this induction schema to $Q$ is what we call Peano Arithmetic, PA.

Here are some important theorems about Peano Arithmetic (PA).

- Gödel’s 1st Incompleteness Theorem: Kurt Gödel proved that with any consistent, finite set of axioms that is powerful enough to do arithmetic, there are always some arithmetical truths that cannot be proved from the axioms. More precisely, he proved that if a statement $X$ asserts its own unprovability in PA, then it is unprovable in PA. This theorem has the following consequence: $X$ must be a tautology in PA. You can see this because $X$ is unprovable (by Gödel’s theorem) and what $X$ states is that $X$ is unprovable. Thus, $X$ states something that must be true. Thus, $X$ is a tautology of PA but unprovable in PA.

- Löb’s Theorem: We know from the fact that PA has a sound proof system that if a statement is provable, it is true. What Löb showed was that if you are able to prove this fact for a particular statement $X$, then you can also prove $X$ directly. That is, if $((X \text{ is provable in PA}) \supset X)$ is provable in PA, then $X$ is provable in PA.

4. Non-standard Models of Arithmetic

Nothing in the axioms of PA (or $Q$) rules out the existence of numbers that cannot be reached by marching along the number line from zero.

PA requires there to be a countable infinity of numbers that are the successors to zero, and these constitute the whole numbers. But there can be extra numbers. Because every number has a successor and every number other than zero has a predecessor, any extra number either need to be embedded in an infinite sequence or a loop of successors. But there can be arbitrarily many infinite sequences of extra numbers and loops of extra numbers.

If we stipulate an additional total ordering, we can rule out the loops. To do so, just introduce a $\leq$-relation holding among all numbers.

1. $\forall x \forall y (x \leq y \lor y \leq x)$
2. $\forall x \forall y ((x \leq y \land y \leq x) \supset x = y)$
3. $\forall x \forall y (x = y \supset x \leq y)$
4. $\forall x \forall y \exists z ((x \leq y \land y \leq z) \supset x \leq z)$
5. $\forall x \forall y (y = s(x) \supset (x \neq y \land y \leq z \land \exists z (z \neq x \land x \leq z \land y \neq z \land z \leq y)))$
6. $\forall x (0 \leq x)$

By doing so, one ensures that the smallest number is zero, that it is followed by all the other whole numbers. Then any extra numbers will be part of some set of extra numbers that is structured like the integers (by being discrete and stretching infinitely in both a positive and negative direction along the number line). There
can be arbitrarily many of these extra integer-like sets, so they can be dense like the rationals and irrationals and they can be “continuous” like the reals.